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Studying conformally flat spacetimes with an elastic stress energy tensor using $1 + 3$ formalism

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ABSTRACT

Conformally flat spacetimes with an elastic stress–energy tensor having diagonal trace-free anisotropic pressure are investigated using $1 + 3$ formalism. The $1 + 3$ Bianchi and Jacobi identities and Einstein field equations are written for a particular case with a conformal factor dependent on only one spatial coordinate. Solutions with non zero anisotropic pressure are obtained.

1. Introduction

The theory of elasticity in the context of general relativity was developed in the mid twentieth century. The need for such a theory came in the late 1950s with Webers bar antenna for gravitational waves [1], in order to explain how these waves interact with elastic solids. Actually, for this phenomena the weak-field approximation was sufficient in the treatment of the problem given by Weber. Only in 1973, in a paper by Carter and Quintana [2], did a fully developed nonlinear theory of elasticity adapted to general relativity appear, remaining to this day a standard reference in the field, although the basic theoretical framework of this theory had already been given by Souriau in [3]. Also before the article by Carter and Quintana, work by Maugin made considerable contributions to the field [4]–[5]. Lately, the theory of elasticity in general relativity was reconsidered by Magli and Kijowski [6]–[7] and Christodoulou [8], in this work they explore the gauge character of relasticity. Authors such as Beig and Schmidt, have proven several existence and uniqueness theorems [9]. More recently, Karlovini and Samuelsson have given a self contained formulation of general relativistic elasticity in [10]. They applied the theory of elasticity to spherically symmetric spacetimes and studied radial and axial perturbations [11–13]. Park [14] established existence theorems for spherically symmetric static solutions for elastic bodies and Brito, Carot and Vaz [15] have obtained static shear-free and non-static shear-free solutions for spherically symmetric elastic spacetimes. Calogero and Heinzle [16] studied the dynamics of Bianchi type I elastic spacetimes.

In this work, we study a very simple elastic model with a diagonal trace-free anisotropy pressure tensor using the $1 + 3$ extended frame approach and following the notation given in Uggla [17]. We note that work on the covariant $1+3$ splitting of fluid spacetime geometries was first initiated by Eisenhart and Synge and continued by Gödel, Raychaudhuri and other authors such as Schöcking, Ehlers, Sachs and Trömper (related Refs. [18–21]). In the paper by Uggla the basic dynamical equations of the extended $1 + 3$ orthonormal frame approach are explicitly given in terms of variables that are naturally adapted to the $1 + 3$ structure, and they include the Bianchi and Jacobi identities, the Einstein field equations and the commutators. This formulation is analogous to the Newmann Penrose approach [22] in the sense that a null congruence is replaced by a timelike congruence. The general properties of the $1 + 3$ orthonormal frame can be found in books such as

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Wald [23] and Felice and Clarke [24] and in Edgar [25]. Several applications have been discussed in Pirani [26], Ellis [27] and MacCallum [28]. A more complete list of references can be found in [17].

The organisation of this paper is as follows. In Section 2 we outline the theory of the 1+3 formalism and present the 1+3 split of the commutators, curvature variables and their field equations, namely Bianchi and Jacobi identities and Einstein field equations. We will follow the same notation convention for tensor indices as used in [17]. Spacetime coordinate tensor indices will be denoted by letters from the second half of the Greek alphabet ($\mu, \nu, \rho, \ldots = 0-3$) while spatial coordinate indices are represented by letters from the second half of the Latin alphabet ($i, j, k, \ldots = 1, \ldots, 3$). Orthonormal frame indices will be denoted by letters from the first half of the Latin alphabet ($a, b, c, \ldots = 0, \ldots, 3$) while spatial frame indices are chosen from the first half of the Greek alphabet ($\alpha, \beta, \gamma, \ldots = 1-3$). In Section 3 we give a summary of the theory of relativistic elasticity. In Section 4 we study conformally flat spacetimes with a conformal factor that depends only on one spatial coordinate and with an elastic source given by a diagonal trace-free anisotropic pressure tensor with only one independent component, using the 1+3 formalism. We calculate the 1+3 equations for a very simple case and determine solutions with non zero anisotropic pressure.

2. 1 + 3 formalism

When studying a dynamical model in general relativity possessing an energy–momentum-stress tensor with a timelike eigendirection, for example perfect fluids, the associated timelike vector field u on the spacetime (M, g) determines the projection tensors U and h, which project parallel and orthogonal to u in the tangent space at each point $p \in (M, g)$, respectively. The vector u is chosen to be a unit timelike vector

$$u_\mu u^\mu = -1, \tag{1}$$

and the projection tensors U and h are defined by

$$U^\mu{}_\nu = -u^\mu u_\nu, \tag{2}$$

$$h^\mu{}_\nu = \delta^\mu{}_\nu + u^\mu u_\nu. \tag{3}$$

Due to the existence of this singled out timelike direction u a covariant 1+3 tensor decomposition of all geometrical objects of physical value can be made using U and h.

∇ will represent the covariant derivative and $\eta^{\mu\nu\rho\sigma}$ the totally antisymmetric permutation tensor.

The well known kinematical fields associated with the timelike congruence u are defined by

$$\dot{u}^\mu = h^\mu{}_\nu u^\rho \nabla_\rho u^\nu, \tag{4}$$

$$\Theta = h^\mu{}_\nu \nabla_\mu u^\nu, \tag{5}$$

$$\sigma_{\mu\nu} = \left[h^\rho{}_{(\mu} h^\sigma{}_{\nu)} - \frac{1}{3} h_{\mu\nu} h^{\rho\sigma} \right] (\nabla_\rho u_\sigma), \tag{6}$$

$$\omega_{\mu\nu} = -h^\rho{}_{[\mu} h^\sigma{}_{\nu]} (\nabla_\rho u_\sigma), \tag{7}$$

where $\dot{u}^\mu$ is the acceleration vector, Θ the rate of expansion scalar, $\sigma_{\mu\nu}$ is the rate of shear tensor and $\omega_{\mu\nu}$ is the vorticity tensor. Note that $\sigma_{\mu\nu}$ is symmetric and tracefree and $\omega_{\mu\nu}$ is anti-symmetric. The magnitude of the rate of shear σ, the vorticity vector ω^μ and the magnitude of the vorticity ω are defined as

$$\sigma^2 = \frac{1}{2} \sigma_{\mu\nu} \sigma^{\mu\nu}, \tag{8}$$

$$\omega^\mu = \frac{1}{2} \eta^{\mu\nu\rho\sigma} \omega_{\nu\rho} u_\sigma, \tag{9}$$

$$\omega^2 = \frac{1}{2} \omega_{\mu\nu} \omega^{\mu\nu} = \omega_\mu \omega^\mu. \tag{10}$$

The vector field u is hypersurface forming if $\omega = 0$.

In the orthonormal frame approach one chooses at each point of the spacetime manifold (M, g) a set of four linearly independent 1-forms e^a such that the line element is given by

$$ds^2 = \eta_{ab} e^a e^b, \tag{11}$$

where $\eta_{ab} = diag[-1, 1, 1, 1]$ is the constant Minkowskian frame metric. The vectors e_a represent the dual basis.

In the 1 + 3 orthonormal frame formalism one aligns the timelike direction of the orthonormal frame with the tangent of the preferred timelike congruence ($e_0 = u$).

2.1. Commutators

The commutator functions are defined by

$$[e_a, e_b] = \gamma^c{}_{ab} e_c, \tag{12}$$