



On Landau–Ginzburg systems, quivers and monodromy



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ABSTRACT

Let X be a toric Fano manifold and denote by $\text{Crit}(f_X) \subset (\mathbb{C}^*)^n$ the solution scheme of the corresponding Landau–Ginzburg system of equations. For toric Del-Pezzo surfaces and various toric Fano threefolds we define a map $L : \text{Crit}(f_X) \rightarrow \text{Pic}(X)$ such that $\mathcal{E}_L(X) := L(\text{Crit}(f_X)) \subset \text{Pic}(X)$ is a full strongly exceptional collection of line bundles. We observe the existence of a natural monodromy map

$$M : \pi_1(L(X) \setminus R_X, f_X) \rightarrow \text{Aut}(\text{Crit}(f_X))$$

where $L(X)$ is the space of all Laurent polynomials whose Newton polytope is equal to the Newton polytope of f_X , the Landau–Ginzburg potential of X , and $R_X \subset L(X)$ is the space of all elements whose corresponding solution scheme is reduced. We show that monodromies of $\text{Crit}(f_X)$ admit non-trivial relations to quiver representations of the exceptional collection $\mathcal{E}_L(X)$. We refer to this property as the M -aligned property of the maps $L : \text{Crit}(f_X) \rightarrow \text{Pic}(X)$. We discuss possible applications of the existence of such M -aligned exceptional maps to various aspects of mirror symmetry of toric Fano manifolds.

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1. Introduction and summary of main results

Let X be a smooth algebraic manifold and let $\mathcal{D}^b(X)$ be the bounded derived category of coherent sheaves on X , see [1,2]. Let A be a finite dimensional associative algebra over the complex numbers and let $\mathcal{D}^b(A)$ be the derived category of right modules over A . One of the fundamental questions in the study of $\mathcal{D}^b(X)$ is the following: *Given a manifold X , is $\mathcal{D}^b(X)$ equivalent to the derived category $\mathcal{D}^b(A)$ of some finite dimensional associative algebra A ?*

The first example of such an equivalence is Beilinson's famous description of $\mathcal{D}^b(X)$ for $X = \mathbb{P}^n$, see [3]. Beilinson shows that $\mathcal{D}^b(\mathbb{P}^n)$ is equivalent to $\mathcal{D}^b(A_n)$ where $A_n = \text{End}(T_n)$ is the endomorphism ring of the vector bundle

$$T_n = \mathcal{O} \oplus \mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(n)$$

In general, an object $E \in \mathcal{D}^b(X)$ is said to be *exceptional* if $\text{Hom}(E, E) = \mathbb{C}$ and $\text{Ext}^i(E, E) = 0$ for $0 < i$. An ordered collection $\{E_1, \dots, E_N\} \subset \mathcal{D}^b(X)$ is said to be an *exceptional collection* if each E_j is exceptional and

$$\text{Ext}^i(E_k, E_j) = 0 \quad \text{for } j < k \text{ and } 0 \leq i$$

An exceptional collection is said to be *strongly exceptional* if also $\text{Ext}^i(E_j, E_k) = 0$ for $j \leq k$ and $0 < i$. A strongly exceptional collection is called *full* if its elements generate $\mathcal{D}^b(X)$ as a triangulated category. In particular, if $\mathcal{E} = \{E_1, \dots, E_N\} \subset \mathcal{D}^b(X)$

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is a full strongly exceptional collection of objects, the corresponding adjoint functors

$$RHom_X(T, -) : \mathcal{D}^b(X) \rightarrow \mathcal{D}^b(A_T); \quad - \otimes_{A_T}^L T : \mathcal{D}^b(A_T) \rightarrow \mathcal{D}^b(X)$$

are equivalences of categories, where $T = \bigoplus_{i=1}^N E_i$. For a given algebraic manifold X one thus asks the following two, related, but not similar questions: (a) does X admit a full exceptional collection of objects in $\mathcal{D}^b(X)$? (b) does X admit a full strongly exceptional collection of line bundles in $Pic(X)$?

A class of manifolds on which these questions have been extensively studied in recent years is the class of toric manifolds and, specifically, the class of toric Fano manifolds, see [4–13]. Question (a) was answered affirmatively by Kawamata which showed that *any toric manifold admits a full exceptional collection of objects in $\mathcal{D}^b(X)$* , see [9]. However, the more refined question (b) of which toric manifolds admit full strongly exceptional collections of line bundles in $Pic(X)$ is currently completely open.

Question (b) has been especially studied for the class of toric Fano manifolds and, indeed, many examples of toric Fano manifolds which admit full strongly exceptional collections have been discovered by various authors. The abundance of examples led experts to ask whether, in fact, any toric Fano manifold admits a full strongly exceptional collection of line bundles in $Pic(X)$, see [4,14]. However, in a recent surprising work [15] Efimov discovered examples of toric Fano manifolds which do not admit any full strongly exceptional collections of line bundles. In particular, the question of which toric Fano manifolds admit full strongly exceptional collections in $Pic(X)$ is currently still open.

On the other hand, the theory of quantum cohomology introduces a family of commutative associative operations $\circ_\omega : H^*(X) \otimes H^*(X) \rightarrow H^*(X)$ parameterized by classes $\omega \in H^*(X)$. This family of “quantum products” defines the structure of a Frobenius super-manifold over $H^*(X)$, which is known as the *big quantum cohomology* of X for instance see, [16]. In particular, the big quantum cohomology is said to be *semi-simple* if the operation \circ_ω is a semi-simple ring operation for generic $\omega \in H^*(X)$. One of the fundamental conjectures on the structure of $\mathcal{D}^b(X)$ is the Dubrovin–Bayer–Manin conjecture, which relates the existence of full exceptional collections of objects in $\mathcal{D}^b(X)$ to the semi-simplicity of the big quantum cohomology of X , see [17,18].

When X is a toric manifold, the Dubrovin–Bayer–Manin conjecture is actually known to hold due to the combined results of Kawamata (on the existence of full exceptional collections of objects in $\mathcal{D}^b(X)$, see [9]) and Iritani (on the semi-simplicity of the big quantum cohomology of toric manifolds, see [19]). In view of the above one is led to ask whether it is possible to relate further, more refined, properties of quantum cohomology to the existence of full strongly exceptional collections of line bundles in $Pic(X)$?

Indeed, of special importance in quantum cohomology theory is the fiber $QH(X) \simeq (H^*(X), \circ_0)$ of the big quantum cohomology over $\omega = 0$, which is known as the *small quantum cohomology* ring of X . When X is a toric Fano manifold the small quantum cohomology is expressed as the Jacobian ring of the Landau–Ginzburg potential, which is a Laurent polynomial $f_X \in \mathbb{C}[z_1^\pm, \dots, z_n^\pm]$ associated to X , see [20–22]. Consider the system of algebraic equations $z_i \frac{\partial}{\partial z_i} f_X(z_1, \dots, z_n) = 0$ for $i = 1, \dots, n$, to which we refer as the Landau–Ginzburg system of equations of X and denote by $Crit(f_X) \subset (\mathbb{C}^*)^n$ the corresponding solution scheme. Our aim in this work is to present, via examples, various relations between properties of the solution scheme $Crit(f_X) \subset (\mathbb{C}^*)^n$ and properties of full strongly exceptional collections of line bundles $\mathcal{E} \subset Pic(X)$. In particular, these relations lead us to suggest a “small variation” of the Dubrovin–Bayer–Manin conjecture for toric Fano manifolds, which we formulate below. As a starting point, consider the following example:

Example (Projective Space). For $X = \mathbb{P}^n$ the Landau–Ginzburg potential is given by $f(z_1, \dots, z_n) = z_1 + \dots + z_n + \frac{1}{z_1 \dots z_n}$ and the corresponding system of equations is

$$z_i \frac{\partial}{\partial z_i} f_X(z_1, \dots, z_n) = z_i - \frac{1}{z_1 \dots z_n} = 0 \quad \text{for } i = 1, \dots, n.$$

The solution scheme $Crit(f_X) \subset (\mathbb{C}^*)^n$ is given by $z_k = (e^{\frac{2\pi ki}{n+1}}, \dots, e^{\frac{2\pi ki}{n+1}})$ for $k = 0, \dots, n$.

In general, Ostrover and Tyomkin show in [22] that X has semi-simple quantum cohomology if and only if the number of elements of $Crit(f_X)$ is $\chi(X)$, the Euler characteristic of X . On the other hand, the expected number of elements in a full strongly exceptional collection in $Pic(X)$ is also $\chi(X)$, see [15]. In view of this, we refer to a map $L : Crit(f_X) \rightarrow Pic(X)$ as an *exceptional map* if its image $\mathcal{E}_L(X) = L(Crit(f_X)) \subset Pic(X)$ is a full strongly exceptional collection. The guiding question is thus the following:

Main Question (small toric Fano DBM-conjecture): Does any toric Fano manifold X whose small quantum cohomology $QH(X)$ is semi-simple admit an exceptional map $L : Crit(f_X) \rightarrow Pic(X)$ naturally generalizing the map $L(z_k) = \mathcal{O}(k)$ in the case of projective space?

Note that defining an exceptional map $L : Crit(f_X) \rightarrow Pic(X)$ requires the association of integral invariants to elements of $Crit(f_X)$. In the case of projective space, such an association is, in fact, misleadingly simple as the entries of the elements are given as roots of unity. However, in general, this is not the case, which is one of the main difficulties in defining the exceptional maps in full generality. Instead, in this work, we consider a few specific examples of toric Fano manifolds, which are known to admit full strongly exceptional collections of line bundles. The manifolds considered are the following: (a) toric Del-Pezzo surfaces, (b) Fano \mathbb{P}^1 -bundles over \mathbb{P}^2 , (c) Fano \mathbb{P}^2 -bundles over \mathbb{P}^1 , (d) \mathbb{P}^1 -bundles over $\mathbb{P}^1 \times \mathbb{P}^1$.

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