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Liberations and twists of real and complex spheres

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ABSTRACT

We study the 10 noncommutative spheres obtained by liberating, twisting, and liberating +twisting the real and complex spheres $S^{N-1}_{\mathbb{R}}, S^{N-1}_{\mathbb{C}}$. At the axiomatic level, we show that, under very strong axioms, these 10 spheres are the only ones. Our main results concern the computation of the quantum isometry groups of these 10 spheres, taken in an affine real/complex sense. We formulate as well a proposal for an extended formalism, comprising 18 spheres.

Introduction

A remarkable discovery, due to Goswami [1], is that each noncommutative compact Riemannian manifold X in the sense of Connes [2–4] has a quantum isometry group $G^+(X)$. While the classical, connected manifolds cannot have genuine quantum isometries [5], for the non-classical or non-connected manifolds the quantum isometry group $G^+(X)$ can be bigger than the usual isometry group $G(X)$, containing therefore "non-classical" symmetries, worth to be investigated.

As a motivating example, the symmetries of the finite noncommutative manifold coming from the Standard Model, axiomatized by Chamseddine and Connes in [6,7], were studied by Bhowmick, D'Andrea, Dabrowski and Das in [8,9]. One of their findings is that the usual gauge group component PU_3 becomes replaced in this way by the quantum group $PU_3^+ = PO_3^+ = \bar{S}_9^+$. Here O_N^+, U_N^+, S_N^+ are the quantum groups constructed by Wang in [10,11], and the twisting result $PO_3^+ = \bar{S}_9^+$ comes from [12].

At a theoretical level, one interesting question is about adapting the various classical computations of isometry groups. Perhaps the most basic such computation is $G(S^{N-1}_{\mathbb{R}}) = O_N$, where $S^{N-1}_{\mathbb{R}} \subset \mathbb{R}^N$ is the standard sphere. Yet another standard computation, this time in the disconnected manifold case, is $G(X_N) = S_N$, where $X_N = \{e_1, \ldots, e_N\} \subset \mathbb{R}^N$ is the simplex, with $e_1, \ldots, e_N$ being the standard basis vectors of $\mathbb{R}^N$.

Such results are of course quite trivial, but their noncommutative extensions, not always. In the discrete manifold case we have $G^+(X_N) = S_N^+$, but more complicated computations, such as $G(Y_N) = H_N$, where $Y_N = \{\pm e_1 \ldots \pm e_N\} \subset \mathbb{R}^N$ is the hypercube, and $H_N = \mathbb{Z}_2 \wr S_N$, lead to some interesting questions. See [13,14].

In the continuous manifold case, which is the one that we are interested in here, the extensions of the basic computation $G(S^{N-1}_{\mathbb{R}}) = O_N$ lead to interesting questions as well. This is well-known for instance in the context of the Podleś spheres [15], and we refer here to [16–18]. More advanced examples of noncommutative spheres, having more intricate algebraic and differential geometry, come from [19,20].

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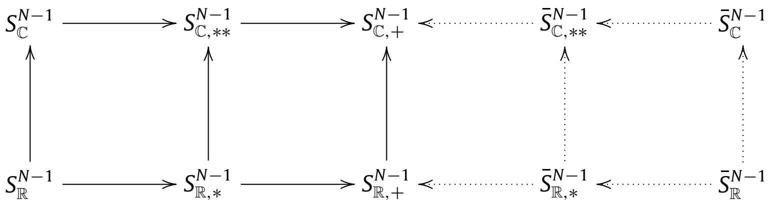
In our joint work with Goswami [21] we introduced two basic generalizations of $S^{N-1}_{\mathbb{R}}$, namely the half-liberated sphere $S^{N-1}_{\mathbb{R},*}$, and the free sphere $S^{N-1}_{\mathbb{R},+}$. These spheres appear by definition as dual objects to certain universal C^*-algebras, inspired by the easy quantum group philosophy [22]. More precisely, the surjections at the C^*-algebra level produce inclusions $S^{N-1}_{\mathbb{R}} \subset S^{N-1}_{\mathbb{R},*} \subset S^{N-1}_{\mathbb{R},+}$, which are related, via the quantum isometry group construction, to the basic inclusions $O_N \subset O_N^* \subset O_N^+$ from [22,23].

Our purpose here is three-fold:

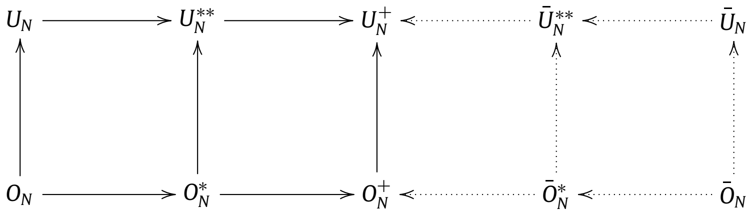
(1) We will review the work in [21], with a new axiomatization of these 3 spheres, less relying on the structure of the corresponding quantum isometry groups.
(2) We will present a unitary extension of [21], based on $G(S^{N-1}_{\mathbb{C}}) = U_N$, with the isometry group being taken in an affine complex sense.
(3) We will present as well a twisting extension of [21], in both the real and complex cases, involving the group $\bar{O}_N$ from [13], and a number of related objects.

We will construct in this way 10 noncommutative spheres, as follows:



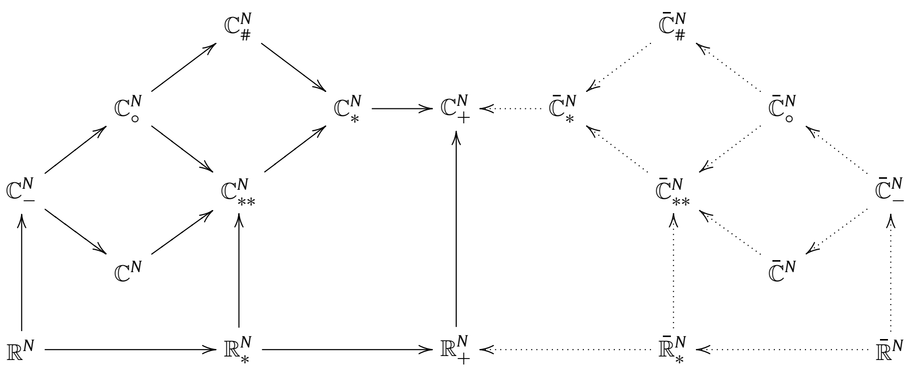
Here all the maps are inclusions. The spheres in [21] are those at bottom left, their complex analogues are on top left, and the whole right part of the diagram appears from the left part via twisting, with the middle spheres being equal to their own twists.

We will prove then that the associated quantum isometry groups, taken in an affine real/complex sense, in the spirit of [14], are as follows:



We believe that our 10 spheres are "smooth" and "Riemannian", in some strong sense, which is yet to be determined. Some questions here, still open, were raised in [21].

At the axiomatic level, we will have results and conjectures stating that, under very strong axioms, our 10 spheres (or "geometries", in a large sense) are the only ones. Our axioms exclude however many interesting objects, like the half-liberated geometry $\mathbb{C}^N_*$ from [8]. Our third contribution will be a proposal, in order to fix this problem. We will show that the 10-geometry formalism has a natural 18-geometry extension, as follows:



Here the geometries $\mathbb{C}^N_- \to \mathbb{C}^N_\circ \to \mathbb{C}^N_\#$ and $\bar{\mathbb{C}}^N_- \to \bar{\mathbb{C}}^N_\circ \to \bar{\mathbb{C}}^N_\#$ are new, and appear when inserting the geometry $\mathbb{C}^N_*$ from [8] and its twist into the 10-geometry framework. This extension, however, requires a lot of work, and we have only partial results here.