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On polynomial integrability of the Euler equations on $\mathfrak{so}(4)$

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ABSTRACT

In this paper we prove that the Euler equations on the Lie algebra $\mathfrak{so}(4)$ with a diagonal quadratic Hamiltonian either satisfy the Manakov condition, or have at most four functionally independent polynomial first integrals.

1. Introduction and statement of the main results

Given a system of ordinary differential equations depending on parameters, it is in general very difficult to recognize for which values of the parameters the equations have first integrals. Except for some simple cases, this problem is very hard and there are no satisfactory methods to solve it.

In this paper we study the first integrals of the Euler differential equations on $\mathfrak{so}(4)$ in $\mathbb{R}^6$ depending on six parameters. Because these equations are used here exclusively as an interesting and nontrivial example of a multiparameter family of ordinary differential equations, we consider them without any explanations of their history, which can be found for instance in the references [1–6]. Here we discuss neither their physical origin, nor their relevant information that can be found in the quoted references.

Before introducing the Euler equations on $\mathfrak{so}(4)$ we first recall some basic definitions on the analytic and polynomial integrability of the polynomial differential systems of the form

$$\frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}} = \mathbf{P}(\mathbf{x}), \quad \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n, \tag{1}$$

with $\mathbf{P}(\mathbf{x}) = (P_1(\mathbf{x}), \dots, P_n(\mathbf{x}))$ and $P_i \in \mathbb{R}[x_1, \dots, x_n]$ for $i = 1, \dots, n$. As usual $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{R}[x_1, \dots, x_n]$ denotes the polynomial ring over $\mathbb{R}$ in the variables $x_1, \dots, x_n$.

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A non-constant function $H(x_1,\dots,x_n)$ is a *first integral* of system (1) on an open subset Ω of $\mathbb{R}^n$ if it is constant on all solution curves $(x_1(t),\dots,x_n(t))$ of system (1) contained in Ω. If H is C^1 on Ω, then H is a first integral of system (1) if and only if

$$\sum_{i=1}^{n} P_i(x)\frac{\partial H}{\partial x_i}(x) \equiv 0, \quad \text{for all } x\in\Omega.$$

If H is a first integral of (1) and is analytic (resp. polynomial), then it is called an *analytic* (resp. a *polynomial*) first integral. An analytic (or formal) first integral $H(x)$ is *reduced* if $H(0)=0$, and for its irreducible decomposition $H(x)=(h_1(x))^{k_1}\dots(h_m(x))^{k_m}$ with $k_i\in\mathbb{N}$ we have $(k_1,\dots,k_m)=1$, i.e. the greatest common factor of $k_1,\dots,k_m$ is one. In what follows, if not specified all analytic (or formal) first integrals are reduced.

The first integrals $H_1,\dots,H_r$ of the differential system (1) with $r<n$ are *functionally independent* if the $r\times n$ matrix

$$\begin{pmatrix} \dfrac{\partial H_1}{\partial x_1} & \dots & \dfrac{\partial H_1}{\partial x_n} \\ \vdots & \dots & \vdots \\ \dfrac{\partial H_r}{\partial x_1} & \dots & \dfrac{\partial H_r}{\partial x_n} \end{pmatrix}(\mathbf{x})$$

has rank r at all points $\mathbf{x}\in\mathbb{R}^n$ where they are defined with the exception (perhaps) of a zero Lebesgue measure set. Moreover, the differential system (1) is *completely integrable* if it has $n-1$ functionally independent first integrals.

The Euler equations on the Lie algebra $\mathfrak{so}(4)$ studied in this paper are

$$\begin{aligned} \frac{dx_1}{dt} &= (\lambda_3-\lambda_2)x_2x_3+(\lambda_6-\lambda_5)x_5x_6 =: P_1(x_1,x_2,x_3,x_4,x_5,x_6),\\ \frac{dx_2}{dt} &= (\lambda_1-\lambda_3)x_1x_3+(\lambda_4-\lambda_6)x_4x_6 =: P_2(x_1,x_2,x_3,x_4,x_5,x_6),\\ \frac{dx_3}{dt} &= (\lambda_2-\lambda_1)x_1x_2+(\lambda_5-\lambda_4)x_4x_5 =: P_3(x_1,x_2,x_3,x_4,x_5,x_6),\\ \frac{dx_4}{dt} &= (\lambda_3-\lambda_5)x_3x_5+(\lambda_6-\lambda_2)x_2x_6 =: P_4(x_1,x_2,x_3,x_4,x_5,x_6),\\ \frac{dx_5}{dt} &= (\lambda_4-\lambda_3)x_3x_4+(\lambda_1-\lambda_6)x_1x_6 =: P_5(x_1,x_2,x_3,x_4,x_5,x_6),\\ \frac{dx_6}{dt} &= (\lambda_2-\lambda_4)x_2x_4+(\lambda_5-\lambda_1)x_1x_5 =: P_6(x_1,x_2,x_3,x_4,x_5,x_6), \end{aligned} \tag{2}$$

defined in $\mathbb{R}^6$. Note that these differential equations depend on six parameters $\lambda_1,\dots,\lambda_6$. We should mention that Eq. (2) are not the most general form of the Euler equation on $\mathfrak{so}(4)$.

It is well known that the Euler equations (2) have always the following three polynomial first integrals of degree 2:

$$H_1=x_1x_4+x_2x_5+x_3x_6, \qquad H_2=\sum_{i=1}^{6}x_i^2, \qquad H_3=\frac{1}{2}\sum_{i=1}^{6}\lambda_i x_i^2,$$

which are functionally independent unless $\lambda_1=\dots=\lambda_6$. We remark that Eq. (2) are Hamiltonian with the Hamiltonian $H=H_3$, and H_1 and H_2 are Casimir functions of the $\mathfrak{so}(4)$ Poisson bracket of the Euler equations.

Recall that a Hamiltonian system in a $2s$-dimensional symplectic manifold is *completely integrable in the Liouvillian sense* if it possesses s functionally independent first integrals which commute with respect to the associated Poisson bracket of the Hamiltonian system. Note that $\mathfrak{so}(4)$ is not a 6-dimensional symplectic manifold, but is a 6-dimensional Poisson manifold foliated into 4-dimensional symplectic leaves by the level sets of the two Casimir functions H_1 and H_2. So there must be two first integrals (the Hamiltonian one and another one) functionally independent on the symplectic leaves for the Euler equations (2) to be Liouvillian integrable on the symplectic leaves.

However, in what follows, when we say that the Euler equations (2) are completely integrable, we always mean that they have five functionally independent first integrals, i.e. the Hamiltonian one and other two ones functionally independent on each symplectic leaf (such systems are usually called superintegrable; recall that a Hamiltonian system on a symplectic manifold M^{2n} is called superintegrable if it has $s>n$ functionally independent first integrals including the Hamiltonian one).

One additional fourth polynomial first integral of degree 2 functionally independent of the first three is known for the Euler equations if the parameters satisfy some conditions, more precisely, either the *product condition* when

$$\lambda_4=\lambda_1, \qquad \lambda_5=\lambda_2, \qquad \lambda_6=\lambda_3,$$

or the *Manakov condition* [7] when

$$M := \lambda_1\lambda_4(\lambda_{23}+\lambda_{56})+\lambda_2\lambda_5(\lambda_{31}+\lambda_{64})+\lambda_3\lambda_6(\lambda_{12}+\lambda_{45})=0. \tag{3}$$