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Causality in noncommutative two-sheeted space-times

Nicolas Franco [a,b,*], Michał Eckstein [c,b]

[a] Copernicus Center for Interdisciplinary Studies, ul. Sławkowska 17, PL-31-016 Kraków, Poland

[b] Faculty of Mathematics and Computer Science, Jagiellonian University, ul. Łojasiewicza 6, PL-30-348 Kraków, Poland

[c] Faculty of Physics, Astronomy and Applied Computer Science, Jagiellonian University, ul. prof. Stanisława Łojasiewicza 11, PL-30-348 Kraków, Poland

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ABSTRACT

We investigate the causal structure of two-sheeted space-times using the tools of Lorentzian spectral triples. We show that the noncommutative geometry of these spaces allows for causal relations between the two sheets. The computation is given in detail when the sheet is a 2- or 4-dimensional globally hyperbolic spin manifold. The conclusions are then generalised to a point-dependent distance between the two sheets resulting from the fluctuations of the Dirac operator.

1. Introduction

Among the pseudo-Riemannian manifolds, the Lorentzian ones form a distinguished class because they can accommodate a causal structure. The latter has very deep consequences for physical models as it sets fundamental restrictions on the evolution of physical processes. On the mathematical side, the causal structure on a Lorentzian manifold $\mathcal{M}$ induces a partial order relation on the set of points of $\mathcal{M}$. The properties of this order have been studied by several authors (see for instance [1–4]).

It turns out that the notion of a partial order can be generalised to the realm of noncommutative spaces [2]. This is to be understood as the existence of a partial order relation on the space of states of a, possibly noncommutative, C^*-algebra A. Via the Gelfand–Naimark theorem, it can be shown that a noncommutative partial order is equivalent to a usual partial order on $\mathrm{Spec}(A)$ whenever A is commutative.

Inspired by these results, we proposed in [5] an extended notion of a causal order suitable for noncommutative geometries (see also [6] for a less formal review). Our definition is embedded in the realm of Lorentzian spectral triples [7] and recovers the classical causal structure for globally hyperbolic spin manifolds [5, Theorem 7]. We note that there exists

* Correspondence to: Department of mathematics, University of Namur, Rempart de la Vierge 8, B-5000 Namur, Belgium.
E-mail addresses: nicolas.franco@math.unamur.be (N. Franco), michal.eckstein@uj.edu.pl (M. Eckstein).

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an alternative approach based on the same ideas [2,8,9], but focusing on more general orders without any specific relation to the metric (so not related to any Dirac operator).

To explore the properties of the proposed noncommutative causal structure we considered in [10] a toy-model based on a noncommutive spectral triple $\left(\mathcal{S}(\mathbb{R}^{1,1})\otimes M_2(\mathbb{C}), L^2(\mathbb{R}^{1,1}, S)\otimes\mathbb{C}^2, \not{D}\otimes 1+\gamma\otimes\mathrm{diag}\{d_1, d_2\}\right)$. It turned out that the triple at hand has a well-defined and highly non-trivial causal structure. It exhibits a number of interesting and unexpected features leading to constraints on the motion not only in the space-time component, but also in the internal space of the model. However, due to the complexity of the computations, we were not able to generalise our results to higher-dimensional, curved, space-times.

In this paper we investigate another toy-model – a two-sheeted space-time – based on a product of a globally hyperbolic space-time $\mathcal{M}$ and a finite spectral triple $\left(\mathcal{A}_F, \mathcal{H}_F, D_F\right)$, with $\mathcal{A}_F = \mathbb{C}\oplus\mathbb{C}$, $\mathcal{H}_F = \mathbb{C}^2$ and $D_F = \begin{pmatrix} 0 & m \\ m^* & 0\end{pmatrix}$. Since the algebra $\mathcal{A}_F$ is a commutative one and has only two pure states, the total space of physical states is isomorphic (at the set-theoretic level) to $\mathcal{M}\sqcup\mathcal{M}$. However, the resulting product geometry is non-trivial because the off-diagonal Dirac operator D_F provides a link between the two sheets.

For this particular model we establish a procedure of determining the causal structure with $\mathcal{M}$ being a general even-dimensional globally hyperbolic manifold. We apply it explicitly in dimensions 2 and 4. Moreover, the adopted technique allows us to generalise the results to the case when the mass parameter m is replaced with a complex scalar field.

The choice of the $\mathbb{C}\oplus\mathbb{C}$ model is also motivated on physical grounds. The noncommutative Standard Model of particle physics, based on the algebra $\mathcal{A}_{SM} = \mathbb{C}\oplus\mathbb{H}\oplus M_3(\mathbb{C})$, is often described as a two-sheeted space-time [11]. Indeed, the space of pure states of the electroweak sector $\mathbb{C}\oplus\mathbb{H}$ consists of two points and although $P(M_3(\mathbb{C}))\cong\mathbb{C}P^2$, all of its points are separated by an infinite distance as the Dirac operator D_F commutes with the $M_3(\mathbb{C})$ part of the algebra [12, Remark 5.1]. Our results on the $M_2(\mathbb{C})$ model suggest that whenever two states are separated by an infinite distance, no causal relation between them is possible. Hence, the chosen finite algebra is a good toy-model for the full Standard Model based on $\mathcal{A}_{SM}$. In this paper we will focus on the mathematical details of the model, postponing the discussion of the physical interpretation to a forthcoming one [13].

The paper is organised as follows: In the next section we recollect the basic definitions and properties of Lorentzian spectral triples and noncommutative causal structures. In Section 3 we describe the features of the two-sheeted model and present the main result of the paper concerning the causal structure. We work out in detail the cases of space-time dimensions 2 and 4 in Sections 4 and 5 respectively. In Section 6 we study the impact of the inner fluctuations of the Dirac operator on the causal structure. We conclude with some general remarks on the applicability of the developed techniques to other almost commutative models.

2. Causality in Lorentzian noncommutative geometry

As a prelude to the introduction of causality in noncommutative geometry, we need to recollect some elements of the theory of Lorentzian spectral triples. The usual definition of a spectral triple, as introduced by Connes [14,15], allows only to deal with (typically compact) Riemannian spaces, while the notion of causality requires non-compact Lorentzian spaces. The first generalisation of the axioms to pseudo-Riemannian signatures was done in [7] and led to various definitions [16–20], which have however a common basis—the Krein space. The theory of pseudo-Riemannian spectral triples is still very recent and undergoes an intense development.

Below we present a rather restrictive definition of a Lorentzian spectral triple following our previous works [6,20]. It has the advantage of guaranteeing a signature of Lorentzian type and allows us to recover a globally hyperbolic spin manifold in the commutative case. The following axioms can also be considered as a particular case of all other existing approaches.

Definition 1. A *Lorentzian spectral triple* is given by the data $(\mathcal{A}, \widetilde{\mathcal{A}}, \mathcal{H}, D, \mathcal{J})$ with:

- A Hilbert space $\mathcal{H}$.
- A non-unital dense $*$-subalgebra $\mathcal{A}$ of a C^*-algebra, with a faithful representation as bounded operators on $\mathcal{H}$.
- A preferred unitisation $\widetilde{\mathcal{A}}$ of $\mathcal{A}$, which is also a dense $*$-subalgebra of a C^*-algebra, with a faithful representation as bounded operators on $\mathcal{H}$ and such that $\mathcal{A}$ is an ideal of $\widetilde{\mathcal{A}}$.
- An unbounded operator D, densely defined on $\mathcal{H}$, such that:
  - $\forall a\in\widetilde{\mathcal{A}}$, $[D, a]$ extends to a bounded operator on $\mathcal{H}$,
  - $\forall a\in\mathcal{A}$, $a(1+\langle D\rangle^2)^{-\frac{1}{2}}$ is compact, with $\langle D\rangle^2 = \frac{1}{2}(DD^*+D^*D)$.
- A bounded operator $\mathcal{J}$ on $\mathcal{H}$ with $\mathcal{J}^2 = 1$, $\mathcal{J}^* = \mathcal{J}$, $[\mathcal{J}, a] = 0$, $\forall a\in\widetilde{\mathcal{A}}$ and such that:
  - $D^* = -\mathcal{J}D\mathcal{J}$ on $\mathrm{Dom}(D) = \mathrm{Dom}(D^*)\subset\mathcal{H}$;
  - there exist a densely defined self-adjoint operator $\mathcal{T}$ with $\mathrm{Dom}(\mathcal{T})\cap\mathrm{Dom}(D)$ dense in $\mathcal{H}$ and with $\left(1+\mathcal{T}^2\right)^{-\frac{1}{2}}\in\widetilde{\mathcal{A}}$, and a positive element $N\in\widetilde{\mathcal{A}}$ such that $\mathcal{J} = -N[D, \mathcal{T}]$.

We say that a Lorentzian spectral triple is *even* if there exists a $\mathbb{Z}_2$-grading γ of $\mathcal{H}$ such that $\gamma^* = \gamma$, $\gamma^2 = 1$, $[\gamma, a] = 0$ $\forall a\in\widetilde{\mathcal{A}}$, $\gamma\mathcal{J} = -\mathcal{J}\gamma$ and $\gamma D = -D\gamma$.

The role of the operator $\mathcal{J}$, called fundamental symmetry, is to turn the Hilbert space $\mathcal{H}$ into a Krein space on which the operator iD is essentially self-adjoint [7,21]. As proved in [6,20], the condition $\mathcal{J} = -N[D, \mathcal{T}]$ guarantees the Lorentzian