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Invariant Ricci collineations on three-dimensional Lie groups

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ABSTRACT

We determine all left-invariant Ricci collineations on three-dimensional Lie groups.

1. Introduction

Symmetries in general relativity have been extensively studied because of their interest both from the mathematical and the physical viewpoint. The term "symmetry" here refers to a one-parameter group of diffeomorphisms of the spacetime preserving certain mathematical or physical quantity. One may regard them as vector fields ξ preserving some tensor field Φ defined on the spacetime like the metric tensor, the curvature tensor, the Ricci tensor or the Weyl tensor. Preserving a geometric quantity is usually understood as the vanishing of the Lie derivative of the geometric quantity in the direction of the vector field, i.e., one has the field equation $\mathcal{L}_\xi \Phi = 0$ on the spacetime. If Φ has geometrical/physical importance, then those special vector fields under which Φ is invariant will also be of importance.

The best known types of symmetries include isometries, homotheties and conformal motions. There exist, however, other types of symmetries which have only been studied more recently: curvature symmetries also called *curvature collineations* (diffeomorphisms which leave the curvature tensor invariant, i.e., $\mathcal{L}_\xi R = 0$), Ricci symmetries also called *Ricci collineations* (diffeomorphisms which preserve the Ricci tensor, i.e., $\mathcal{L}_\xi \operatorname{Ric} = 0$), *Weyl collineations* (diffeomorphisms which preserve the Weyl tensor, i.e., $\mathcal{L}_\xi W = 0$), etc., which are often more difficult to deal with than those mentioned above (see [1–3] for more information and further references). Similarly, a *matter collineation* is a vector field ξ whose flow preserves the energy–momentum tensor $T = \operatorname{Ric} - \frac{1}{2}\tau g$ (equivalently, $\mathcal{L}_\xi T = 0$). In many cases of interest, the set of all such symmetries can be given a Lie group structure whose generators span a Lie algebra of vector fields.

Ricci collineations present interesting geometric links with the study of infinitesimal concircular transformations and have been used to determine the underlying structure of the spacetime. For instance, any conformal Ricci collineation (i.e., ξ is a conformal vector field and a Ricci collineation simultaneously) is either homothetic or the underlying metric is a Brinkmann wave provided that ξ and the gradient of $\operatorname{div} \xi$ are orthogonal [4]. Ricci and curvature collineations have been

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classified in many physically interesting spacetimes [5–9] and moreover, their properties can also be used to determine the structure of those spacetimes (see [10,1,11,12] and the previously mentioned references).

The purpose of this paper is to determine all left-invariant Ricci collineations on three-dimensional Lie groups. Since the connection and Ricci tensor are built from the metric tensor, it must inherit its symmetries. Hence every Killing vector field is an affine Killing vector field and a Ricci collineation but the converse may not be true, so we will emphasize the existence of the non-affine Killing ones; for this purpose we will take advantage of the subspaces of Killing and affine Killing vector fields which have been determined in [13]. Also, note that any homothetic vector field is a Ricci collineation (and hence so is any Yamabe soliton with constant scalar curvature [14]). Hence, we will focus on the *proper* case, i.e., Ricci collineations which are neither Killing nor homothetic.

We organize this paper as follows. In Section 2 we review the description of all three-dimensional Lorentzian Lie algebras and fix some basic notational conventions. We analyze the existence of left-invariant Ricci collineations on unimodular Lorentzian Lie groups in Section 3, while the non-unimodular Lorentzian case is considered in Section 4. In each case we determine the corresponding vector subspace of Ricci collineations, and therefore it is possible to compare it with the subspaces of Killing and affine Killing left-invariant vector fields obtained in [13]. Finally, in Section 5 the Riemannian case is analyzed; we will omit the details since the results are obtained essentially as in Sections 3 and 4. Moreover, in Section 5 it is also shown that the class of left-invariant Ricci collineations is strictly larger than the class of Lorentzian Yamabe solitons.

2. Preliminaries

2.1. Three-dimensional Lorentzian Lie algebras

Let $\times$ denote the Lorentzian vector product on $\mathbb{R}^3_1$ induced by the product of the para-quaternions (i.e., $e_1 \times e_2 = -e_3$, $e_2 \times e_3 = e_1$, $e_3 \times e_1 = e_2$, where $\{e_1, e_2, e_3\}$ is an orthonormal basis of signature $(++-)$). Then $[Z, Y] = L(Z \times Y)$ defines a Lie algebra, which is unimodular if and only if L is a self-adjoint endomorphism of $\mathfrak{g}$ [15]. Considering the different Jordan normal forms of L, we have the following four classes of unimodular three-dimensional Lorentzian Lie algebras.

Type Ia. If L is diagonalizable with eigenvalues $\{\alpha, \beta, \gamma\}$ with respect to an orthonormal basis $\{e_1, e_2, e_3\}$ of signature $(++-)$, then the corresponding Lie algebra is given by

$$[e_1, e_2] = -\gamma e_3, \qquad [e_1, e_3] = -\beta e_2, \qquad [e_2, e_3] = \alpha e_1. \tag{$\mathfrak{g}_{Ia}$}$$

Type Ib. Assume L has a complex eigenvalue. Then, with respect to an orthonormal basis $\{e_1, e_2, e_3\}$ of signature $(++-)$, one has

$$L = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \gamma & -\beta \\ 0 & \beta & \gamma \end{pmatrix}, \quad \beta \neq 0$$

and thus the corresponding Lie algebra is given by

$$[e_1, e_2] = \beta e_2 - \gamma e_3, \qquad [e_1, e_3] = -\gamma e_2 - \beta e_3, \qquad [e_2, e_3] = \alpha e_1. \tag{$\mathfrak{g}_{Ib}$}$$

Type II. Assume L has a double root of its minimal polynomial. Then, with respect to an orthonormal basis $\{e_1, e_2, e_3\}$ of signature $(++-)$, one has

$$L = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \frac{1}{2} + \beta & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} + \beta \end{pmatrix}$$

and thus the corresponding Lie algebra is given by

$$[e_1, e_2] = \frac{1}{2} e_2 - \left(\beta - \frac{1}{2}\right) e_3, \qquad [e_1, e_3] = -\left(\beta + \frac{1}{2}\right) e_2 - \frac{1}{2} e_3, \qquad [e_2, e_3] = \alpha e_1. \tag{$\mathfrak{g}_{II}$}$$

Type III. Assume L has a triple root of its minimal polynomial. Then, with respect to an orthonormal basis $\{e_1, e_2, e_3\}$ of signature $(++-)$, one has

$$L = \begin{pmatrix} \alpha & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \alpha & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \alpha \end{pmatrix}$$