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Causal and conformal structures of two-dimensional globally hyperbolic spacetimes

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ABSTRACT

The group of conformal diffeomorphisms and the group of causal automorphisms on two-dimensional globally hyperbolic spacetimes are clarified. It is shown that if two-dimensional spacetimes have non-compact Cauchy surfaces, then the groups are subgroups of that of two-dimensional Minkowski spacetime, and if two-dimensional spacetimes have compact Cauchy surfaces, then the groups are subgroups of that of two-dimensional Einstein's static universe. Also, the groups of such spacetimes are explicitly calculated by use of universal covering spaces.

1. Introduction

Liouville's Theorem states that there are some kind of rigidity on conformal structures of semi-Euclidean space $\mathbb{R}^n_\nu$ when $n \geq 3$. In other words, any conformal diffeomorphisms defined on an open subset U of $\mathbb{R}^n_\nu$ are generated by homotheties, isometries and inversions [1–4]. Since inversion has singularity, to study conformal structure, we need conformal compactifications [5,6].

In contrast to this, in two-dimensional Euclidean space, it is known that any conformal diffeomorphisms defined on an open subset of $\mathbb{R}^2$ are homography or anti-homography and these can be seen as a conformal map defined on Riemann sphere [5].

In this paper, causal structures and conformal structures of two-dimensional globally hyperbolic spacetimes are analyzed. Though some authors introduce conformal compactification of two-dimensional Minkowski spacetime, if we confine the subject to spacetimes with Cauchy surfaces, we can explicitly obtain their groups of conformal diffeomorphisms without compactifications. It is known that the group of conformal diffeomorphisms can be obtained by the group of causal automorphisms if the dimension of the Lorentzian manifold is bigger than two [7–9] and so, in high dimensional Lorentzian manifolds to study conformal structures is equivalent to study causal structures. However, this is not the case for two-dimensional spacetimes.

For this reason, to study conformal or causal structure of two-dimensional spacetimes has sufficient meanings and so, in this paper, we study coherently both causal and conformal structures of two-dimensional spacetimes with Cauchy surfaces by tools developed in Section 3. One of the main results is that if two-dimensional spacetimes have non-compact Cauchy surfaces, then their structure groups are subgroups of that of two-dimensional Minkowski spacetime $\mathbb{R}^2_1$, and if two-dimensional

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spacetimes have compact Cauchy surfaces, then their structure groups are subgroups of that of two-dimensional Einstein's static universe E. In this sense, $\mathbb{R}^2_1$ and E play the role of co-universal objects among two-dimensional globally hyperbolic spacetimes.

2. Preliminaries

A Lorentzian manifold is a differentiable manifold with the signature of metric as $(-, +, \ldots, +)$. A tangent vector $v \in T_pM$ is called timelike, null and spacelike if $g(v, v)$ is less than 0, equal to 0 and greater than 0, respectively. We say that a tangent vector is causal if it is timelike or null. It is easy to see that the set of all causal vectors has two connected components and we choose one of them to be future-directed vectors and the other to be past-directed vectors. It is a well-known fact that a differentiable manifold M has a Lorentzian metric if and only if M has a nowhere-vanishing vector field X. This nowhere-vanishing vector field can be used to define a time-orientation which determines future-directed vectors. We remark that not all Lorentzian manifolds are time-orientable. (See p. 2 of Ref. [10] or Proposition 37 of Chapter 5 in [11].) Throughout this paper, we are interested in those Lorentzian manifolds which are time-oriented. By spacetime, we mean a Lorentzian manifold with time-orientation.

We denote by $x \leq y$ if there exists a continuous curve γ from x to y such that for each t, there exists a neighborhood U of $\gamma(t)$ such that for any $\gamma(t_1)$ and $\gamma(t_2)$ in U with $t_1 < t_2$, there exists a smooth curve $\alpha : [0, 1] \to U$ from $\gamma(t_1)$ to $\gamma(t_2)$, of which the tangent vector $\alpha'(t)$ is future-directed and causal for all t. When $x \leq y$ we say that x and y are causally related or y lies in the future of x. By use of convex normal neighborhood, it can be shown that $x \leq y$ if and only if there exists a piecewise differentiable curve γ such that $\gamma'(t)$ is future-directed and causal for each t.

A bijective map $f : M \to N$ between two Lorentzian manifolds is called a causal isomorphism (anti-causal isomorphism, respectively) if f satisfies the condition that $x \leq y$ if and only if $f(x) \leq f(y)$ ($f(x) \geq f(y)$, respectively). When the domain of definition and the codomain coincide, we call the causal isomorphism as a causal automorphism.

It turns out that causal relation of a Lorentzian manifold has close relations to conformal structure of the given manifold. In 1964, Zeeman has shown that any causal isomorphism on n-dimensional Minkowski spacetime $\mathbb{R}^n_1$ is generated by homothety and isometries if $n \geq 3$ [12]. In 1976, Hawking et al. had shown that if a spacetime is strongly causal, causal isomorphism becomes a smooth conformal diffeomorphism [8]. In 1977, Malament had shown that causal isomorphism on any spacetime is a smooth conformal diffeomorphism [9]. However, as authors commented, their results do not hold for $n = 2$. In two-dimensional Minkowski spacetime, there are many more continuous causal isomorphisms [13,14].

3. Causal structure and covering space

Given a covering map $\pi : \overline{M} \to M$ where M is a semi-Riemannian manifold with metric g, we define the metric of $\overline{M}$ by use of pull-back $\overline{g} = \pi^* g$. Then π is a smooth local isometry. When M is a Lorentzian manifold, we define a time-orientation on $\overline{M}$ in such a way that π is a time-orientation preserving local isometry. To be precise, if a vector field X^a defines a time-orientation on M, then the pull-back 1-form $\pi^* X_a$ can be used to define the time-orientation on $\overline{M}$.

A diffeomorphism $\phi : \overline{M} \to \overline{M}$ is called a covering transformation if $\pi \circ \phi = \pi$. It is easy to see that the set of all covering transformations of $\pi : \overline{M} \to M$ forms a group and we denote it by D.

Theorem 3.1. *Let $\pi_M : \overline{M} \to M$ and $\pi_N : \overline{N} \to N$ be universal covering maps of spacetimes. If $f : M \to N$ is a causal isomorphism, then any lift of $f \circ \pi_M$ through π_N is a causal isomorphism.*

Proof. Choose $x \in M$ and $\overline{x} \in \pi_M^{-1}(x)$. Let $y = f(x)$ and choose $\overline{y} \in \pi_N^{-1}(y)$.

Since $\overline{M}$ is simply-connected, we can lift $f \circ \pi_M$ through π_N and so we get a map $\overline{f} : (\overline{M}, \overline{x}) \to (\overline{N}, \overline{y})$ that satisfies $\pi_N \circ \overline{f} = f \circ \pi_M$.

Since $\overline{N}$ is simply-connected, we can lift $f^{-1} \circ \pi_N$ through π_M and so we get a map $\overline{f^{-1}} : (\overline{N}, \overline{y}) \to (\overline{M}, \overline{x})$ that satisfies $\pi_M \circ \overline{f^{-1}} = f^{-1} \circ \pi_N$.

By combining the above two equalities, we have $\pi_M \circ \overline{f^{-1}} \circ \overline{f} = \pi_M$ and $\pi_N \circ \overline{f} \circ \overline{f^{-1}} = \pi_N$.

Since $\overline{f^{-1}} \circ \overline{f}(\overline{x}) = \overline{x}$ and $\overline{f} \circ \overline{f^{-1}}(\overline{y}) = \overline{y}$, by the uniqueness of lifts, we must have $\overline{f^{-1}} \circ \overline{f} = Id_{\overline{M}}$ and $\overline{f} \circ \overline{f^{-1}} = Id_{\overline{N}}$. Therefore, $\overline{f}$ is a bijection from $\overline{M}$ to $\overline{N}$ with its inverse $(\overline{f})^{-1} = \overline{f^{-1}}$.

We now show that $\overline{f}$ is a causal isomorphism. Choose $\overline{x_1}$ and $\overline{x_2}$ in $\overline{M}$ such that $\overline{x_1} \leq \overline{x_2}$ and let $\overline{\alpha}$ be a future-directed causal curve from $\overline{x_1}$ to $\overline{x_2}$. Then, since π_M is a time-orientation preserving local isometry and f is a causal isomorphism, the curve $f \circ \pi_M \circ \overline{\alpha}$ is a future-directed causal curve in N.

Since $\pi_N \circ \overline{f} = f \circ \pi_M$, $\overline{f} \circ \overline{\alpha}$ is the lift of the causal curve $f \circ \pi_M \circ \overline{\alpha}$ through π_N. Since π_N is a time-orientation preserving local isometry, $\overline{f} \circ \overline{\alpha}$ is a future-directed causal curve and thus we have $\overline{f}(\overline{x_1}) \leq \overline{f}(\overline{x_2})$.

By applying the same argument for $\overline{f^{-1}}$ and f^{-1}, we can show that $\overline{x_1} \leq \overline{x_2}$ if and only if $\overline{f}(\overline{x_1}) \leq \overline{f}(\overline{x_2})$ and so $\overline{f}$ is a causal isomorphism. □

Under some mild conditions, any causal isomorphisms between two Lorentzian manifolds are smooth conformal diffeomorphisms [7–9]. However, this is not the case when the dimension of Lorentzian manifold is two, and thus we need to prove the previous theorem in topological terms [12–14].

On the other hand, if we assume sufficient smoothness, we can prove the following.