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Homogeneous generalized holomorphic structures

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Homogeneous generalized holomorphic structures in the context of homogeneous principal fiber bundles are investigated. They are characterized in terms of Lie algebra data, and the generalized Dolbeault cohomology groups associated to a homogeneous generalized holomorphic vector bundle are identified with certain relative Lie algebra cohomology groups. We also provide some examples, using generalized flag manifolds.

1. Introduction

Generalized complex (GC for short) geometry is a common format for complex geometry and symplectic geometry, initiated to set up a framework for certain topics in string theory.

Generalized holomorphic (GH for short) vector bundles are the generalized complex analogue of holomorphic vector bundles in classical complex geometry, including flat bundles over symplectic manifolds, co-Higgs bundles and holomorphic Poisson modules as extreme examples. These examples are also the most studied cases up to date. For some details concerning these examples, see [1–5].

However, to construct more general GH vector bundles is not that easy. Instead of considering GH vector bundles, the author has introduced the notion of GH structures in the context of generalized principal fiber bundles in [6], and developed a related deformation theory in [7]. This paper then is a continuation of those two papers, motivated by the attempt to find more general examples.

One way to construct principal fiber bundles is through the theory of homogeneous spaces, which often provides good examples for testing geometrical ideas; meanwhile, invariant GC structures on homogeneous manifolds have already been studied by B. Milburn [8]. Based on these, this paper is then devoted to considering homogeneous principal fiber bundles and trying to characterize *homogeneous GH structures* (see Section 3 for the precise notion and structure) in terms of Lie algebra data. It turns out that this really provides a convenient method of constructing GH structures.

In the deformation theory of [7], when a given GH structure is deformed, the underlying GC structure is also allowed to vary together with the GH structure. To apply this theory successfully, a preliminary knowledge of the generalized Dolbeault cohomology associated to the initial GC structure and GH structure is, more or less, necessary. Therefore, the generalized Dolbeault cohomology associated to a homogeneous GH vector bundle is also investigated in the context of the theory of Lie

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algebra cohomology. This may provide an algebraic way to compute the relevant cohomology groups in some special cases in the future.

The paper is organized as follows. In Section 2, we recall the necessary preliminaries of generalized (complex) geometry. Although we will mainly deal with a special kind of exact Courant algebroids—namely the generalized tangent bundles of manifolds without a twist, we still start from a more general setting for our later convenience. The following sections are devoted to K-homogeneous GH structures for K a connected Lie group: In Section 3, we characterize a homogeneous GH structure in terms of Lie algebra data (cf. Theorems 3.2 and 3.3). In Section 4, we prove that the generalized Dolbeault cohomology associated to a homogeneous GH vector bundle is equivalent to certain relative Lie algebra cohomology (cf. Theorem 4.1). When K is compact, this allows a canonical decomposition of the cohomology groups as K-modules (cf. Theorem 4.2). With the aid of the preceding results, Section 5 produces some examples of homogeneous GH structures using generalized flag manifolds. The decomposition mentioned above is also illustrated by two very simple examples.

2. Backgrounds

We collect the necessary basics of generalized geometry in this section. The main references are [9–11,6]. Throughout the paper, M will always be a connected orientable smooth manifold.

Generalized geometry is the geometry related to the generalized tangent bundle $\mathbb{T}M := TM \oplus T^*M$, or more generally, a so-called exact Courant algebroid E.

Definition 2.1. An exact Courant algebroid over M is a real vector bundle $E \to M$ with a bracket $[\cdot,\cdot]_c$ (Courant bracket) on $\Gamma(E)$, a nondegenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$, and an anchor map $\pi : E \to TM$, satisfying the following conditions for all $e_1, e_2, e_3 \in \Gamma(E)$ and $f \in C^\infty(M)$:

- $\pi([e_1, e_2]_c) = [\pi(e_1), \pi(e_2)]$,
- $[e_1, [e_2, e_3]_c]_c = [[e_1, e_2]_c, e_3]_c + [e_2, [e_1, e_3]_c]_c$,
- $[e_1, fe_2]_c = f[e_1, e_2]_c + (\pi(e_1)f)e_2$,
- $\pi(e_1)\langle e_2, e_3\rangle = \langle [e_1, e_2]_c, e_3\rangle + \langle e_2, [e_1, e_3]_c\rangle$,
- $[e_1, e_1]_c = \frac{1}{2}\mathcal{D}\langle e_1, e_1\rangle$,
- $0 \longrightarrow T^*M \xrightarrow{\pi^*} E \xrightarrow{\pi} TM \longrightarrow 0$ is exact,

where $\mathcal{D} = \pi^* \circ d : C^\infty(M) \to \Gamma(E)$ and E^*, E are identified using $\langle\cdot,\cdot\rangle$.

Given an exact Courant algebroid E, one can find an isotropic splitting $s : TM \to E$, which has a curvature form $H \in \Omega^3_{cl}(M)$ defined by

$$H(X, Y, Z) = \langle [s(X), s(Y)]_c, s(Z)\rangle, \quad X, Y, Z \in \Gamma(TM).$$

By the bundle isomorphism $s + \pi^* : TM \oplus T^*M \to E$, the Courant algebroid structure can be transported onto $\mathbb{T}M$. Then the pairing $\langle\cdot,\cdot\rangle$ is the natural one, i.e. $\langle X + \xi, Y + \eta\rangle = \xi(Y) + \eta(X)$, and the Courant bracket is

$$[X + \xi, Y + \eta]_H = [X, Y] + L_X\eta - \iota_Y d\xi + \iota_Y\iota_X H,$$

called the H-twisted Courant bracket. E has more symmetries than TM. Besides the usual diffeomorphisms preserving the cohomology class $[H]$, a 2-form b also gives rise to an automorphism of E called a B-transform: $e^b(A) = A + \iota_{\pi(A)}b$ for $A \in \Gamma(E)$.

An isotropic subbundle of E is called a generalized distribution and called integrable if it is involutive under the Courant bracket. An integrable maximal generalized distribution is called a Dirac structure. These notions can be complexified and what interests us is the following complex Dirac structure[1]:

Definition 2.2. A GC structure in E is an orthogonal complex structure $\mathbb{J}$ of E, such that the i-eigenbundle $L \subset E_{\mathbb{C}}$ of $\mathbb{J}$ is integrable. We call $(M, \mathbb{J})$ a GC manifold.

A GC manifold is always of even dimension, say, $2m$. Two extreme GC structures are symplectic and complex structures. Let $E = \mathbb{T}M$ with $H = 0$. If ω is a symplectic structure, then $L = \{X - i\omega(X) | X \in T_{\mathbb{C}}M\}$; if J is a complex structure, then $L = T_{0,1} \oplus T^*_{1,0}$. A more complicated example is a holomorphic Poisson manifold (M, J, β), for which, $L = \{X + \xi + \beta(\xi) | X + \xi \in T_{0,1} \oplus T^*_{1,0}\}$.

We will use $\mathbb{J}$ and L interchangeably to label the GC structure under consideration. By the pairing, L^* can be identified with $\bar{L}$, and a differential $d_L : \Gamma(\wedge^k \bar{L}) \to \Gamma(\wedge^{k+1}\bar{L})$ can be defined: for $\sigma \in \Gamma(\wedge^k \bar{L})$, $a_i \in \Gamma(L)$,

$$\begin{aligned} d_L\sigma(a_0, \ldots, a_k) &= \sum_i (-1)^i \pi(a_i)\sigma(a_0, \ldots, \widehat{a_i}, \ldots, a_k) \\ &\quad + \sum_{i<j} (-1)^{i+j}\sigma([a_i, a_j]_c, a_0, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_k), \end{aligned} \tag{2.1}$$

[1] We use $V_{\mathbb{C}}$ to denote the complexification of a real vector space or bundle V.