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Affine distributions on Riemannian manifolds with applications to dissipative dynamics

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ABSTRACT

Using a coordinate free characterization of hyperplanes intersection, we provide explicitly a set of local generators for a smooth affine distribution given by those smooth vector fields $X \in \mathfrak{X}(U)$ defined eventually on an open subset $U \subseteq M$ of a smooth Riemannian manifold (M, g), that verifies the relations $g(X, X_1) = \cdots = g(X, X_k) = 0, g(X, Y_1) = h_1, \ldots, g(X, Y_p) = h_p$, where $X_1, \ldots, X_k, Y_1, \ldots, Y_p \in \mathfrak{X}(U)$, and respectively $h_1, \ldots, h_p \in \mathcal{C}^\infty(U, \mathbb{R})$, are a-priori given quantities.

In the case when $X_1, \ldots, X_k, Y_1, \ldots, Y_p$ are gradient vector fields associated with some smooth functions $I_1, \ldots, I_k, D_1, \ldots, D_p \in \mathcal{C}^\infty(U, \mathbb{R})$, i.e., $X_1 = \nabla_g I_1, \ldots, X_k = \nabla_g I_k, Y_1 = \nabla_g D_1, \ldots, Y_p = \nabla_g D_p$, then we obtain a set of local generators for the smooth affine distribution of smooth vector fields which conserve the quantities $I_1, \ldots, I_k$ and dissipate the scalar quantities $D_1, \ldots, D_p$ with prescribed rates $h_1, \ldots, h_p$.

1. Introduction

The theory of distributions, in the broadest sense, it is perhaps one of the most influential tool from differential geometry, according to its usefulness in a large assortment of scientific domains, e.g., geometric control theory, differential equations, sub-Riemannian geometry, dynamical systems, nonholonomic mechanics (for details see, e.g., [1–6]).

The main protagonist of this paper is a special class of distributions, namely the so called smooth affine distributions on Riemannian manifolds. More precisely, the main purpose of this work is to provide explicitly (and coordinate free) a set of local generators for a given smooth affine distribution on a finite dimensional smooth Riemannian manifold. Moreover, by applying this result to some special classes of affine distributions, we generalize some results from [7] related to smooth affine distributions associated to dissipative dynamical systems.

More exactly, in the second section we provide a coordinate free formulation for the intersection of a finite number of hyperplanes of a finite dimensional inner product space (Euclidean vector space).

The third section is dedicated to the local study of smooth affine distributions on Riemannian manifolds. Since the main applications of our results are supposed to improve the study of conservative/dissipative dynamical systems, for practical reasons, we will focus on the local study of smooth affine distributions. More precisely, the main purpose of this section is to provide explicitly (and coordinate free) a set of local generators for a smooth affine distribution given by those vector fields $X \in \mathfrak{X}(U)$ defined eventually on an open subset $U \subseteq M$ of a smooth Riemannian manifold (M, g), that verifies the relations $g(X, X_1) = \cdots = g(X, X_k) = 0, g(X, Y_1) = h_1, \ldots, g(X, Y_p) = h_p$, where $X_1, \ldots, X_k, Y_1, \ldots, Y_p \in \mathfrak{X}(U)$, and respectively $h_1, \ldots, h_p \in \mathcal{C}^\infty(U, \mathbb{R})$ are a-priori given quantities. The analysis of mixed homogeneous and nonhomogeneous relations

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is deliberate (even if one can recover the homogeneous part by simply annihilating the nonhomogeneous one) because of the clarity of formulas for the local generators. Moreover, these two classes of relations have completely different meaning in dynamical setting, as can be seen in the next section.

The aim of the last section is to apply the results obtained in the previous section to some special classes of smooth affine distributions naturally associated to dynamical systems, and also to provide a unified presentation of conservative and dissipative dynamical systems. More exactly, in the case when $X_1, \dots, X_k, Y_1, \dots, Y_p$ are gradient vector fields on a smooth Riemannian manifold (M, g) associated with some smooth functions $I_1, \dots, I_k, D_1, \dots, D_p \in \mathcal{C}^\infty(U, \mathbb{R})$, i.e., $X_1 = \nabla_g I_1, \dots, X_k = \nabla_g I_k, Y_1 = \nabla_g D_1, \dots, Y_p = \nabla_g D_p$, then we obtain a set of local generators for the smooth affine distribution of vector fields which conserve the quantities $I_1, \dots, I_k$ and dissipate the scalar quantities $D_1, \dots, D_p$ with prescribed rates $h_1, \dots, h_p$. As a consequence one obtains a generalization for $p > 1$ of a result from [7] given for $p = 1$. Note that for $p > 0$ the main result provides a local characterization of dissipative dynamical systems. Some other dynamically relevant cases are obtained, e.g., for $p = 0$ one obtains a local characterization of conservative dynamical systems; for $p = 0$ and $k = \dim M - 1$ one obtains a local characterization of completely integrable dynamical systems.

2. A coordinate free formulation of hyperplanes intersection

In this section we obtain a coordinate free formulation for the linear variety determined by the intersection of a finite number of hyperplanes of a finite dimensional inner product space.

In order to do that, let us recall that given an n-dimensional inner product space $(E, \langle\cdot,\cdot\rangle)$ over a field $\mathbb{K}$ of characteristic zero, then for any $p \in \{1, \dots, n\}$, the pth exterior power of the vector space E, $\Lambda^p E$, inherits an inner product, $\langle\cdot,\cdot\rangle_p$, defined on pairs of decomposable p-vectors by

$$\langle v_1 \wedge \cdots \wedge v_p, w_1 \wedge \cdots \wedge w_p\rangle_p := \det[\langle v_i, w_j\rangle_{1\leq i,j\leq p}], \tag{2.1}$$

for any $v_1 \wedge \cdots \wedge v_p, w_1 \wedge \cdots \wedge w_p \in \Lambda^p E$, and extended by bilinearity to the whole vector space $\Lambda^p E$. Note that $(\Lambda^1 E, \langle\cdot,\cdot\rangle_1) = (E, \langle\cdot,\cdot\rangle)$, and by convention $\Lambda^0 E = \mathbb{K}$.

As usual, one denotes by $\|\cdot\|_p = \sqrt{\langle\cdot,\cdot\rangle_p}$, the norm induced by the inner product $\langle\cdot,\cdot\rangle_p$. Note that in the case of the inner product $\langle\cdot,\cdot\rangle_p$,

$$\|v_1 \wedge \cdots \wedge v_p\|_p = \sqrt{\det(G(v_1, \dots, v_p))},$$

where $G(v_1, \dots, v_p) = [\langle v_i, v_j\rangle_{1\leq i,j\leq p}]$ is the Gram matrix associated to the ordered set of vectors $\{v_1, \dots, v_p\} \subset E$.

Recall that any orthonormal basis of E, $\{e_1, \dots, e_n\}$, generates an orthonormal basis of $\Lambda^p E$,

$$\{e_{i_1} \wedge \cdots \wedge e_{i_p} | 1 \leq i_1 \leq \cdots \leq i_p \leq n\}.$$

Consequently, for any $p \in \{0, \dots, n\}$, we have

$$\dim_{\mathbb{K}} \Lambda^p E = \binom{n}{p} = \binom{n}{n-p} = \dim_{\mathbb{K}} \Lambda^{n-p} E,$$

and hence $\Lambda^p E \cong \Lambda^{n-p} E$. A natural isomorphism between these vector spaces is given by the Hodge star operator.

In order to remind the definition of the Hodge star operator let us fix an orthonormal basis of the vector space E, say $\{e_1, \dots, e_n\}$, and the corresponding basis unit vector $\mu = e_1 \wedge \cdots \wedge e_n$ for the vector space $\Lambda^n E$. Note that since the basis $\{e_1, \dots, e_n\}$ is orthonormal, we get that $\|\mu\|_n = 1$. The volume element μ is hence unique up to a sign and defines an orientation of E.

Fixing $p \in \{1, \dots, n\}$ and $\nu \in \Lambda^p E$, we get that the map

$$\omega \in \Lambda^{n-p} E \mapsto \nu \wedge \omega \in \Lambda^n E,$$

is linear, and hence there exists a unique linear functional $\alpha_\nu \in \left(\Lambda^{n-p} E\right)^\star$ such that

$$\nu \wedge \omega = \alpha_\nu(\omega)\mu.$$

Since $\Lambda^{n-p} E$ is an inner product space, due to Riesz representation of linear functionals, we have the existence of a unique element of $\Lambda^{n-p} E$, denoted $\star\nu$, such that for any $\omega \in \Lambda^{n-p} E$,

$$\alpha_\nu(\omega) = \langle\star\nu, \omega\rangle_{n-p}.$$

Hence, for any $\nu \in \Lambda^p E$ and $\omega \in \Lambda^{n-p} E$, we have

$$\langle\star\nu, \omega\rangle_{n-p} \quad \mu = \nu \wedge \omega. \tag{2.2}$$

Summarizing, the linear operator $\nu \in \Lambda^p E \mapsto \star\nu \in \Lambda^{n-p} E$, is by construction an isomorphism of vector spaces, and is called the Hodge star operator. Moreover, a direct consequence of the property (2.2) and of the fact that $(\star \circ \star)(\nu)$