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Coherent state map quantization in a Hermitian-like setting

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ABSTRACT

For vector bundles having an involution on the base space, Hermitian-like structures are defined in terms of such an involution. We prove a universality theorem for suitable self-involutive reproducing kernels on Hermitian-like vector bundles. This result relies on pullback operations involving the tautological bundle on the Grassmann manifold of a Hilbert space and exhibits the aforementioned reproducing kernels as pullbacks of universal reproducing kernels that live on the Hermitian-like tautological bundle. To this end we use a certain type of classifying morphisms, which are geometric versions of the coherent state maps from quantum theory. As a consequence of that theorem, we obtain some differential geometric properties of these reproducing kernels in this setting.

1. Introduction

This paper belongs to a line of research [1–5] on differential geometric aspects of reproducing kernels and their related structures. As explained below, the present development was suggested by some problems that claim their origin in the broad interaction between the reproducing kernels and the quantum theory, more precisely in the applications of the coherent state method such as it is proposed and carried out in [6–9], among other papers.

Coherent state maps

By coherent state map we mean a smooth (symplectic) map $\zeta\colon Z \to \mathbb{CP}(\mathcal{H})$ of a (finite-dimensional Kähler) manifold Z into a complex projective Hilbert space $\mathbb{CP}(\mathcal{H})$, for some complex Hilbert space $\mathcal{H}$. (Recall that $\mathbb{CP}(\mathcal{H})$ may well be viewed as the Grassmannian manifold on $\mathcal{H}$ formed by the one-dimensional subspaces of $\mathcal{H}$.) In usual physical interpretations, Z is to be regarded as the classical phase space of a mechanical system and $\mathbb{CP}(\mathcal{H})$ as the space of pure quantum states. The transition probability amplitudes in Z can be expressed in terms of a reproducing kernel K defined on the line bundle $\mathbb{L} \to Z$ obtained as the pullback of the tautological bundle $\mathbb{E} \to \mathbb{CP}(\mathcal{H})$, through the map ζ. Indeed, the existence of the kernel K

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is equivalent to the existence of the map ζ, and this fact can be formulated in terms of appropriate equivalent categories. In this way, the (Fubini–Study) Hermitian and complex structures on $\mathbb{E} \to \mathbb{CP}(\mathcal{H})$ induce corresponding Hermitian and complex structures, and associated Chern covariant derivative, on the bundle $\mathbb{L} \to Z$; see [6,7]. The area of application of the coherent state method has been widened to abstract settings of quantization involving either polarized C^*-algebras [8] or positive kernels on bundles acted on by compact groups [9]. In the latter case the space $\mathbb{CP}(\mathcal{H})$ is replaced by the n-Grassmannian on $\mathcal{H}$ formed by all the n-dimensional subspaces of $\mathcal{H}$, with fixed $n \in \mathbb{N}$. For links between coherent state maps and algebraic geometry, see [10,11], and also [12]. Other applications of reproducing kernels in physics can be seen in [13,14], for example.

Primary motivation of the present research

Our aim has been the study of quantization problems related to the above circle of ideas, in an infinite-dimensional setting, from the perspective of operator theory or operator algebras and their physical implications. The mathematical framework of this approach is provided by differential geometry of infinite-dimensional manifolds and Banach–Lie groups acting on these manifolds. In this connection, some references related in spirit to the present investigation are [15–18], [19], and also the measure-free approach of [20], since the absence of suitable measures for defining L^2-spaces is one of the difficulties encountered in infinite dimensions, which can be addressed by using reproducing kernel Hilbert spaces instead. It is worth mentioning that techniques from this area of infinite-dimensional geometry also turned out to be relevant for applications in quantum chemistry [21], the study of Berry's geometrical phase factor [22], quantum optics [23], etc.

One motivation for the proposed study relies on the general observation that the notions involved in the coherent state method have significant physical meaning in infinite dimensions. See for example [24–26] and references therein, for infinite-dimensional Grassmannians. More specific and direct motivation for the present paper comes from the geometric models of Borel–Weil type for representations of unitary groups of C^*-algebras, which were obtained in [27] by using reproducing kernels on suitable Hermitian homogeneous vector bundles. Such kernels yield the representation Hilbert spaces in the geometric models. In order to include holomorphy and full groups of invertible elements of C^*-algebras in that theory, the notion of like-Hermitian vector bundle was introduced in [1] in close relationship with the existence of involutions on the base spaces of those bundles and with the corresponding complexifications. An approach to these topics in the framework of category theory was developed in [3], where several universality results were established which allow us to recover the reproducing kernels from the tautological ones associated with universal Grassmann vector bundles. In this respect, the above-mentioned projective manifold $\mathbb{CP}(\mathcal{H})$ is substituted by the full Grassmannian manifold $\mathrm{Gr}(\mathcal{H})$ on $\mathcal{H}$, the line bundle $\mathbb{E} \to \mathbb{CP}(\mathcal{H})$ by the tautological bundle $\mathcal{T}(\mathcal{H}) \to \mathrm{Gr}(\mathcal{H})$ and the coherent state map $\zeta: Z \to \mathbb{CP}(\mathcal{H})$ by a classifying morphism $\zeta: Z \to \mathrm{Gr}(\mathcal{H})$; (it is to be noticed that infinite-dimensional Grassmannian manifolds are usually considered in physics as classical phase spaces, whereas they play here the role of a quantization tool) see [3, Sect. 5].

Technical aspects: universality and geometry of reproducing kernels

On the other hand, reproducing kernels and connections or covariant derivatives and associated geometric objects often occur simultaneously on vector bundles, as it happens in the setting worked on in [6,7], for instance. Thus it sounds sensible to find out possible explanations for such an occurrence in general. In the Hermitian case, we have recently approached that question by relying on the basic idea of transferring the geometry of the tautological bundle $\mathcal{T}(\mathcal{H}) \to \mathrm{Gr}(\mathcal{H})$ to the given bundle, say $D \to Z$, on the basis of the universality theorems and corresponding classifying morphisms. See [4] for the construction of the natural connection induced by a given reproducing kernel K, and the calculation of the corresponding covariant derivative ∇_K. It is shown in [5] that ∇_K is essentially the Chern covariant derivative for $D \to Z$; that is, the connection which is compatible both with the Hermitian and the complex structure (if there is one) in $D \to Z$. Some possible links with algebraic geometry are also implicit in [5].

As is well known, complex structures are very important both in quantization of mechanical systems and in Borel–Weil realizations. Some examples of the homogeneous spaces and vector bundles considered in [27], in relation with unitary representations, are indeed holomorphic, but this is not true in general. In order to include holomorphy and *full* groups of invertible elements of C^*-algebras in the theory of [27], the notion of like-Hermitian vector bundle was introduced and studied in [1] in close relationship with the existence of involutions on the base spaces of those bundles and with the corresponding complexifications. Moreover, the like-Hermitian bundles have been discussed from a categorial viewpoint in [3]. However, the universality theorem for reproducing kernels on like-Hermitian bundles, given in [3, Th. 5.1], is not entirely satisfactory in the sense that its proof required a somehow unpleasant *additional assumption* in the statement of the theorem —namely, that the classifying morphism should commute *a priori* with the involutions on the base spaces of the bundles under consideration. Since dealing with involutions in the base spaces of vector bundles is essential to introduce holomorphy in the theory via the complexifications (so is essential to the level of generality, examples and applications given in [1] in particular) one would like to find an improvement of [3, Th. 5.1] which in turn allows us to transfer geometry from the tautological bundles to bundles with involution.

The present paper is not intended as a merely formal generalization of results of [3] or [4]. For the sake of precision, one must insist that the purpose of this paper is two-fold. Firstly, we give an improvement of [3, Th. 5.1] which seems to be the right theorem of universality for bundles endowed with an involutive structure (see Theorem 4.1). Then a new