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# On the structure and volume growth of submanifolds in Riemannian manifolds

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ABSTRACT

The aim of this note is two-fold. First, we investigate the relations between the volume growth of a submanifold and its second fundamental form. In the second part, we discuss the relations between the index of some Schrödinger operators and the structure of a submanifold, and prove some one-end theorems.



## 1. Introduction

The metric and topological structure of complete submanifolds in various ambient spaces has been studied extensively during past few years. In [1], Cao–Shen–Zhu proved that a complete immersed stable minimal hypersurface $M^n$ in $\mathbb{R}^{n+1}$ with $n \geq 3$ must have only one end. This result was generalized by Li–Wang [2], they showed that a complete, oriented, minimal hypersurface with finite index in $\mathbb{R}^{n+1}$ must have finitely many ends. In [3], Yun proved that for a complete oriented minimal hypersurface $M^n$ of $\mathbb{R}^{n+1}$, $n \geq 3$, if the $L^n$-norm of its second fundamental form is less than an explicit constant, then there are no $L^2$-harmonic 1-forms on $M$, which implies that $M$ has only one end. Fu–Li [4] obtained that if an oriented complete submanifold $M^n$ $(n \geq 3)$ in $\mathbb{R}^{n+m}$ has finite total mean curvature, i.e., $\int_M |H|^n dv < \infty$, and the $L^n$-norm of its traceless second fundamental form is less than an explicit constant, then $M$ has only one end.

As well known, the volume growth plays an important role in the study of the structure of a non-compact submanifold. In [5–7], it shows that there is a significant relation between the number of ends and the behavior of a quotient of volumes. Hence another issue we explore here is about the influence of the second fundamental form of a Riemannian submanifold on its volume growth. There are many references on this subject, one can consult [8,9], etc.

Before stating our main results, let us fix some notations. We say that a complete Riemannian manifold $N^{n+m}$ has non-negative $(n-1)$th Ricci curvature if for any $x \in N$ and any $n$ orthonormal vectors $\{e, e_1, \ldots, e_{n-1}\} \subset T_xN$, the curvature tensor satisfies $\sum_{i=1}^{n-1} \langle R(e_i, e)e, e_i\rangle \geq 0$.

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Let $M^n$ be a complete non-compact submanifold in a Riemannian manifold $N^{n+m}$. Fix a point $x \in M$ and a local orthonormal frame $\{e_1, \dots, e_{n+m}\}$ of $N^{n+m}$ such that $\{e_1, \dots, e_n\}$ are tangent fields of $M^n$ at $x$. Recall that the second fundamental form $A$ is defined by

$$A = \sum_{\alpha=n+1}^{n+m} A^\alpha e_\alpha$$

with $A^\alpha_{ij} = \langle \overline{\nabla}_{e_i} e_\alpha, e_j \rangle$, where $\overline{\nabla}$ is the Riemannian connection of $N^{n+m}$. The squared norm $|A|^2$ of the second fundamental form and the mean curvature vector $H$ are defined respectively by:

$$|A|^2 = \sum_{\alpha=n+1}^{n+m} \sum_{i,j=1}^{n} (A^\alpha_{ij})^2 \quad \text{and} \quad H = H^\alpha e_\alpha = \frac{1}{n} \sum_{\alpha=n+1}^{n+m} tr(A_\alpha) e_\alpha.$$

Recall that the traceless part of the second fundamental form is defined by

$$\phi = \sum_{\alpha=n+1}^{n+m} \phi^\alpha e_\alpha$$

where $\phi^\alpha_{ij} = A^\alpha_{ij} - H^\alpha g_{ij}$. Obviously, the tensor $\phi$ satisfies

$$|\phi|^2 = \sum_{\alpha=n+1}^{n+m} \sum_{i,j=1}^{n} (\phi^\alpha_{ij})^2 = |A|^2 - n|H|^2.$$

In this paper, we shall propose to study the structure of complete submanifolds. By the results of Li–Tam [10], the parabolic or non-parabolic property of a complete manifold is related to its volume growth. Hence, we will first estimate the volume growth of a submanifold $M^n$ under various extrinsic curvature bounds. In particular, we will consider a situation in which the second fundamental form of $M^n$ satisfies some $L^p$-integrability or pointwise conditions, and describe the upper bound of its volume growth.

**Theorem 1.1.** *Let $M^n$ be a complete non-compact submanifold immersed in a complete Riemannian manifold of non-negative $(n-1)$-th Ricci curvature. Suppose that*

$$\int_{B(r)} |\phi|^p dv = \circ(r^\alpha) \quad as\ r \to \infty$$

*for some $\alpha > 0$ and $p > n$. Then*

$$Vol(B(r)) = \circ(r^{p+\alpha}) \quad as\ r \to \infty$$

*and $\lambda_1(M) = 0$. In particular, if $\int_M |\phi|^p dv < \infty$ for some $p > n$, then $\lambda_1(M) = 0$.*

Finally, we will investigate the topology at infinity of a complete submanifold in a Cartan–Hadamard manifold, i.e., a complete, simply connected manifold with non-positive sectional curvature. This will be done by using harmonic functions. More specifically, we first prove non-parabolicity of each end of the submanifold under a spectral assumption on a relevant Schrödinger operator, and then show the following one end theorem by using the theory of harmonic functions.

**Theorem 1.2.** *Let $M^n$, $n > 2$, be a complete non-compact immersed submanifold in the Euclidean space $\mathbb{R}^{n+m}$. If $\lambda_1(\Delta + \frac{\sqrt{n-1}}{2}|A|^2) \geq 0$, then $M$ has only one end.*

## 2. Preliminaries

The structure of a complete manifold is closely related to the lower bound of its Ricci curvature. The proofs of our main results are based on the lower bounds of the Ricci curvature, which can be estimated in appropriate form in terms of the second fundamental form of submanifolds.

**Lemma 2.1.** *Let $M^n$ $(n \geq 2)$ be a submanifold immersed in a Riemannian manifold $N^{n+m}$ with non-negative $(n-1)$th Ricci curvature. Then,*

(i) *the Ricci curvature of $M^n$ satisfies*

$$\mathrm{Ric}_M \geq -\frac{n}{4}|\phi|^2,$$

*and*

$$\mathrm{Ric}_M \geq -\frac{\sqrt{n-1}}{2}|A|^2.$$