Journal of Geometry and Physics 92 (2015) 140–146

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Maximal surface equation on a Riemannian 2-manifold with finite total curvature

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ARTICLE INFO

Article history:
Received 6 May 2014
Received in revised form 28 January 2015
Accepted 13 February 2015
Available online 20 February 2015

MSC:
58J05
53C42
53C50

Keywords:
Maximal surface equation
Finite total curvature
Lorentzian geometry

ABSTRACT

The differential equation of maximal surfaces on a complete Riemannian 2-manifold with finite total curvature is studied. Uniqueness theorems that widely extend the classical Calabi–Bernstein's theorem in non-parametric version, as well as previous results on complete maximal graphs into Lorentzian warped products, are given. All entire solutions of maximal equation in certain natural Lorentzian warped product, as well as non-existence results, are provided.

1. Introduction

The classical Calabi–Bernstein's theorem for maximal surfaces in the 3-dimensional Lorentz–Minkowski space $\mathbb{L}^3$, in non-parametric version, states that the only entire solutions to the maximal surface equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right)=0, \quad |Du|<1 \tag{1}$$

on the Euclidean plane $\mathbb{R}^2$ are affine functions.

A maximal surface in $\mathbb{L}^3$ is a spacelike surface with zero mean curvature. The term spacelike means that the induced metric from the ambient Lorentzian metric is a Riemannian metric on the surface. The terminology maximal comes from a variational problem, since these surfaces locally maximize area among all nearby surfaces having the same boundary. Besides their mathematical interest, maximal surfaces and, more generally, spacelike surfaces with constant mean curvature are also important in General Relativity (see, for instance, [1]).

A singular fact in Lorentzian products (or warped products) in contrast to the case of graph into complete Riemannian products (or warped products) is that an entire spacelike graph in the Lorentzian case is not necessarily complete, in the sense that the induced Riemannian metric is not necessarily complete (see [2, Section 4]).

The theorem aforementioned is a relevant uniqueness result, which was first proved by Calabi [3] and later extended for maximal hypersurfaces in $\mathbb{L}^n$ by Cheng and Yau [4]. It can also be stated in terms of the local complex representation of the surface [5,6]. Even two different types of direct simple proofs are given in [7] and [8].

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http://dx.doi.org/10.1016/j.geomphys.2015.02.011

In [9], the authors give new examples of non-parametric Calabi–Bernstein type problems for warped Lorentzian products, whose warping function is non-locally constant and its fiber is the Euclidean plane. Obviously the Calabi–Bernstein theorem is not included in this case. A new version of non-parametric Calabi–Bernstein type theorem in the case of a Lorentzian product $\mathbb{R}\times F$, where F denotes a Riemannian 2-manifold, with non-negative curvature and positive at some point, has been given in [2] and [10]. Recently, another Calabi–Bernstein type results in the more general ambient of a warped Lorentzian product are given in [11] and [12].

In this work we deal with entire solutions of the maximal surface equation on a complete 2-dimensional Riemannian manifold with finite total curvature. Recall that a complete Riemannian surface has finite total curvature if the integral of the absolute value of its Gaussian curvature is finite (Section 2.4). In fact, consider the following nonlinear elliptic differential equation, in divergence form:

$$\operatorname{div}\left(\frac{Du}{f(u)\sqrt{f(u)^2-|Du|^2}}\right)=-\frac{f'(u)}{\sqrt{f(u)^2-|Du|^2}}\left(2+\frac{|Du|^2}{f(u)^2}\right) \tag{E.1}$$

$$|Du|<f(u) \tag{E.2}$$

where f is a smooth real-valued function defined on an open interval I of the real line $\mathbb{R}$, the unknown u is a function defined on a domain Ω of a non-compact complete Riemannian surface (F,g) with finite total curvature, $u(\Omega)\subseteq I$, D and div denote the gradient and the divergence of (F,g) and $|Du|^2:=g(Du,Du)$. The constraint (E.2) is the ellipticity condition. We are mainly interested in uniqueness and non-existence results for entire solutions (i.e. defined on all F) of Eq. (E).

The solutions of (E) are the extremals under interior variations for the functional

$$u\longmapsto\int f(u)\sqrt{f(u)^2-|Du|^2}\,dA,$$

where dA is the area element for the Riemannian metric g, which acts on functions u such that $u(\Omega)\subseteq I$ and $|Du|<f(u)$.

This variational problem naturally arise from Lorentzian geometry. In order to see this, consider the product manifold $M:=I\times F$ endowed with the Lorentzian metric

$$\langle\,,\,\rangle=-\pi_I^*(dt^2)+f(\pi_I)^2\pi_F^*(g), \tag{2}$$

where π_I and π_F denote the projections from M onto I and F, respectively. The Lorentzian manifold $(M=I\times_f F,\langle\,,\,\rangle)$ is a warped product, in the sense of [13, p. 204], with base $(I,-dt^2)$, fiber (F,g) and warping function f. Any warped product $I\times_f F$ possesses an infinitesimal timelike conformal symmetry (see Section 2.1) which is an important tool in this paper.

For each $u\in C^\infty(\Omega)$, $u(\Omega)\subseteq I$, the induced metric on Ω from the Lorentzian metric (2), via its graph $\Sigma_u=\{(u(p),p):p\in\Omega\}$ in M, is written as follows:

$$g_u=-du^2+f(u)^2g,$$

and it is positive definite, i.e. Riemannian, if and only if u satisfies $|Du|<f(u)$ everywhere on Ω. When g_u is Riemannian, $f(u)\sqrt{f(u)^2-|Du|^2}\,dA$ is the area element of (Ω,g_u). Therefore (E.1) of (E) is the Euler–Lagrange equation for the area functional, its solutions are spacelike graphs of zero mean curvature in M, and this equation is called the maximal surface equation in M.

If we denote by N the unit normal vector field N on Σ_u such that $\langle N,\partial_t\rangle\geq 1$ on Σ_u, where $\partial_t:=\partial/\partial t\in\mathfrak{X}(M)$, then

$$N=\frac{-f(u)}{\sqrt{f(u)^2-|Du|^2}}\left(1,\frac{1}{f(u)^2}Du\right),$$

and the hyperbolic angle θ between $-\partial_t$ and N is given by

$$\langle N,\partial_t\rangle=\cosh\theta=\frac{f(u)}{\sqrt{f(u)^2-|Du|^2}}.$$

Observe that when $I=\mathbb{R}$, $F=\mathbb{R}^2$ and $f=1$, Eq. (E) is the maximal surface equation in $\mathbb{L}^3$. Of course, the Euclidean plane $\mathbb{R}^2$ has finite total curvature, but note that any complete Riemannian surface whose curvature is non-negative out a compact set has finite total curvature (see Section 2.4). On the other hand, examples of complete minimal surfaces in $\mathbb{R}^3$ with finite total curvature are known (see, [14]). Examples in a different ambient space can be seen in [15].

In this work, we give new results on uniqueness and non-existence of solutions of Eq. (E) on a complete Riemannian surface with finite total curvature. Our results widely extend and improve the non-parametric following results, [9, Th. A and Th. B], [2, Th. 4.3 and Cor. 4.4], [10, Cor. 8], [11, Th. 6.2 and Th. 6.3] and [12, Th. 4.2 and Cor. 4.3]. Moreover, Calabi–Bernstein's Theorem is included as a particular case (see Section 4).