Journal of Geometry and Physics 92 (2015) 181–193

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

The Kepler system as a reduced 4D harmonic oscillator

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ARTICLE INFO

Article history:
Received 23 December 2013
Received in revised form 16 December 2014
Accepted 20 February 2015
Available online 27 February 2015

MSC:
primary 53D20
37J15
70H05
70H33

Keywords:
Symplectic reduction
Harmonic oscillator
Kepler problem

ABSTRACT

In this paper we review the connection between the Kepler problem and the harmonic oscillator. More specifically we consider how the Kepler system can be obtained through geometric reduction of the harmonic oscillator. We use the method of constructive geometric reduction and explicitly construct the reduction map in terms of invariants. The Kepler system is obtained in a particular chart on the reduced phase space. This reduction is the reverse of the well known KS regularization. Furthermore the reduced phase space connects to Moser's regularization. The integrals for the Kepler system given by the momentum and Laplace vectors, as well as the Delaunay elements, can now be easily related to symmetries of the harmonic oscillator.

1. Introduction

In this note we will review the relation between the Kepler problem and the isotropic 4 DOF harmonic oscillator. This is done from the view point of reduction of the harmonic oscillator. As is well known the Kepler problem refers to the bounded motion of a particle in $\mathbb{R}^3$ which is influenced by the gravitational field of a second particle fixed at the origin. Because there is a vast literature about this problem we will not try to be complete in our references and only refer to literature that provides the ingredients for this paper.

The Kepler problem has the disadvantage that it has singularities corresponding to collision orbits. This problem can be overcome by regularizing the problem, i.e. constructing a transformation which transforms the Kepler system into a system for which the solutions exist for all time. The oldest method is due to Levi-Civita who regularized the two-dimensional Kepler system [1–3]. This was generalized to three dimensions by Kustaanheimo and Stiefele [4,5]. Another regularization is due to Moser [6]. All these regularizations map bounded orbits of the Kepler system to orbits of the harmonic oscillator. The KS-regularization relates the Kepler system to a harmonic oscillator on $\mathbb{R}^8$. Moser regularization relates the Kepler system to a constrained harmonic oscillator on $T^+S^3 \subset \mathbb{R}^8$. Here T^+S^3 is the tangent bundle to the unit sphere minus its zero section. In [7] it is shown what the relation is between these two regularizations. Also the Ligon–Schaaf map [8] can be used to regularize the Kepler problem. We will not deal with this in this paper. More details can be found in [9,10]. The Ligon–Schaaf regularization can be seen as an adaption of the Moser regularization (see [11]).

The idea to relate the Kepler system to the harmonic oscillator by means of reduction is not new. The main ideas can be found in papers of Kummer [7,12,13], where the relation between the KS- and Moser regularization is actually shown by using reduction of the harmonic oscillator. Related ideas are found in [14]. Also in the paper by D'Avanzo and Marmo [15] the idea of reduction is used. In this paper actually the Kepler system is embedded in a higher dimensional system by what

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http://dx.doi.org/10.1016/j.geomphys.2015.02.016
0393-0440/

is called unfolding. This unfolding is a reverse reduction. They use the KS regularization to unfold to the harmonic oscillator, thus establishing a relation between KS regularization and reduction.

In this paper we will consider geometric reduction of the harmonic oscillator. It can be seen as a review of Kummer's results in the more modern context of reduction through orbit maps. We will use constructive geometric reduction which is based on the geometric reduction as introduced by Meyer [16] and Marsden and Weinstein [17]. The more constructive approach in [18], extending the earlier results to the singular case, gives a general framework for the construction of reduced phase spaces as introduced in [19–21]. The reduced phase space is constructed by considering an orbit map in terms of invariants for the symmetry. In general the reduced phase space is obtained as a semi algebraic set. The symplectic form on the original phase space induces a Poisson structure on the orbit space which restricts to a symplectic form on the reduced phase space (see also [10,22]). A slightly different approach to reduction is used in [15] using methods from [23]. The orbit map approach has the advantage that everything becomes explicitly computable in terms of invariants and Poisson brackets.

In the case of the harmonic oscillator reduction is performed with respect to a specific S^1-action generated by a quadratic integral $\varXi$. More precisely the Hamiltonian of the harmonic oscillator is $H_2(q,Q) = \frac{1}{2}|q|^2 + \frac{1}{2}|Q|^2$, $(q,Q) \in \mathbb{R}^8$, and $\varXi(q,q) = (q_1Q_2 - q_2Q_1) + (q_3Q_4 - q_4Q_3)$. The result is a 3 DOF system on the reduced phase space. The Kepler system is then obtained by reparametrizing the orbits of the reduced harmonic oscillator in a particular chart for the reduced phase space. This way a reconstruction (unfolding in [15]) of the orbits of the reduced system map the orbits of the Kepler system to orbits of the harmonic oscillator.

The methods can also be used to relate integrals of the harmonic oscillator to integrals of the Kepler problem. As the orbit map is a Poisson map and conserves the commutation relations. This way the symmetries of both systems can be easily related because any symplectic symmetry group for the system reduces to a symplectic symmetry group on the reduced phase space.

The reduction of the harmonic oscillator to obtain the Kepler system can be seen as a mere observation. However, this reduction can also be applied to perturbations of the harmonic oscillator which are symmetric with respect to the action generated by $\varXi$. This way perturbations of the harmonic oscillator can be reduced to perturbations of the Kepler system, a remark already made in [13]. Note that the perturbations of the Kepler system that can be obtained this way are only those perturbations that can be regularized. We shall conclude this introduction with some remarks but not deal with the subject of perturbed Keplerian systems in this paper. Regularized perturbed Keplerian systems can be considered as perturbed isotropic harmonic oscillators on $\mathbb{R}^8$, which can be described by a Hamiltonian systems with the standard symplectic form, and co-ordinates (q, Q). Symmetry can be imposed by normalization up to a certain order in the perturbation and truncation of higher order terms. Consider a perturbed isotropic oscillator on $\mathbb{R}^8$

$$H(q,Q) = \frac{1}{2}(Q,Q) + \frac{1}{2}(q,q) + \varepsilon\mathcal{H}(q,Q)$$

with additional symmetry given by the integral

$$\varXi(q,Q) = q_1Q_2 - Q_1q_2 + q_3Q_4 - Q_3q_4. \tag{1}$$

When such a system is in normal form with respect to its quadratic part H_2 one may reduce with respect the $\varXi$ symmetry to obtain a three degree of freedom system and further reduce with respect to the H_2 symmetry to a two degree of freedom system with, for $\varXi = 0$, a reduced phase space $S^2 \times S^2$ with Poisson structure equivalent to that of the Lie algebra $\mathfrak{so}(3)\times\mathfrak{so}(3)$. This reduced phase space obtained after two reductions of the harmonic oscillator corresponds to the reduced phase space obtained for the Kepler system after one reduction, as the reduced harmonic oscillator for $\varXi = 0$, after one reduction, corresponds to the Kepler system. Thus one can consider these systems as generalizations of perturbed Keplerian systems. Systems like this with additional integral $L_1(q,Q) = -q_1Q_2 + Q_1q_2 + q_3Q_4 - Q_3q_4$ are studied in detail in [24–27]. The results in this paper provide proof for the fact that the systems in these papers are actual generalizations of perturbed Keplerian systems, such as the Zeeman problem and the Van der Waals problem in [26]. Some more remarks on how regularized perturbed Keplerian system lead to perturbed harmonic oscillators can be found in [28]. Perturbed Keplerian systems that can be studied in the framework of this paper are the lunar problem [13,29], the main problem of artificial satellite theory [30], the orbiting dust problem [29].

The set up of the paper is as follows. In Section 2 we review the Kepler system and KS and Moser regularization within its classical context. In Section 3 we consider the harmonic oscillator and its geometric reduction. In Section 4 we show how the Kepler system is obtained as a reduced system for the harmonic oscillator by formulating the reduced system in a proper chart on the reduced phase space. The chart is given by the KS regularization, and the reduced phase space is T^+S^3, which is the space on which Moser regularizes the Kepler problem. In Section 5 we show how reduction of the Kepler problem fits in further reduction of the harmonic oscillator. Here the integrals of the Kepler problem are represented as integrals on the harmonic oscillator level. In Section 6 we make some remarks on Delaunay variables.

2. The Kepler system and regularization

We may describe the Kepler system as a Hamiltonian system $(\tilde{K}, T_0\mathbb{R}^3, \tilde{\omega})$, with phase space $T_0\mathbb{R}^3 = (\mathbb{R}^3 - \{0\}) \times \mathbb{R}^3$, with co-ordinates (x, y), and symplectic form $\tilde{\omega} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + dx_3 \wedge dy_3$. The Hamiltonian is given by

$$\tilde{K}(x,y) = \frac{1}{2}\langle y,y\rangle - \frac{\mu}{|x|}.$$