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Quantum automorphism groups of finite quantum groups are classical

Paweł Kasprzak[a], Piotr M. Sołtan[a,*], Stanisław L. Woronowicz[b,a]

[a] Department of Mathematical Methods in Physics, Faculty of Physics, University of Warsaw, Poland

[b] Institute of Mathematics, University of Bialystok, Poland

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ABSTRACT

In a recent paper of Bhowmick, Skalski and Sołtan the notion of a quantum group of automorphisms of a finite quantum group was introduced and, for a given finite quantum group $\mathbb{G}$, existence of the universal quantum group acting on $\mathbb{G}$ by automorphisms was proved. We show that this universal quantum group is in fact a classical group. The key ingredient of the proof is the use of multiplicative unitary operators, and we include a thorough discussion of this notion in the context of finite quantum groups.

1. Finite quantum groups and associated multiplicative unitaries

In the literature on non-commutative geometry and quantum groups the definition of a quantum group is still not established. Some authors prefer the point of view that a quantum group is nothing else than a Hopf algebra or an appropriate topological analog of a Hopf algebra. Others make a point of explicitly invoking the duality between spaces and commutative algebras and say that a quantum group is an object $\mathbb{G}$ of the category dual to that of a certain class of algebras (say C*-algebras) with additional properties. These properties are only expressible in therms of the object $\mathbb{G}$ viewed back in the category of algebras and written as e.g. $\mathrm{C}(\mathbb{G})$. Then the fact that $\mathbb{G}$ is a "quantum group" amounts to a Hopf algebra-like structure on $\mathrm{C}(\mathbb{G})$.

In this paper we will try to stay on middle ground between these approaches. A *finite quantum group* will simply be a finite-dimensional Hopf $*$-algebra $(\mathscr{A}, \Delta)$ over $\mathbb{C}$ such that $\mathscr{A}$ admits a faithful positive functional. In this situation $\mathscr{A}$ is in fact a C*-algebra, so in particular it is semisimple. On the other hand the compact quantum group of automorphisms of $(\mathscr{A}, \Delta)$ will be denoted by $\mathbb{H}$ and the notation will be that of e.g. [1,2].

The *Haar functional* or the *Haar measure* of $(\mathscr{A}, \Delta)$ is a normalized positive linear functional $\boldsymbol{h}$ on $\mathscr{A}$ such that

$$(\boldsymbol{h} \otimes \mathrm{id})\Delta(a) = (\mathrm{id} \otimes \boldsymbol{h})\Delta(a) = \boldsymbol{h}(a)\mathbb{1}, \quad a \in \mathscr{A}.$$

* Corresponding author.
E-mail addresses: pawel.kasprzak@fuw.edu.pl (P. Kasprzak), piotr.soltan@fuw.edu.pl (P.M. Sołtan), stanislaw.woronowicz@fuw.edu.pl (S.L. Woronowicz).

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Existence of $\boldsymbol{h}$ is a classical result in Hopf algebras [3,4]. The functional $\boldsymbol{h}$ is faithful [5] and consequently introduces a scalar product on $\mathscr{A}$ via

$$\langle a|b\rangle = \boldsymbol{h}(a^*b), \quad a, b \in \mathscr{A}. \tag{1.1}$$

It is clear from formula (1.1) that we adopt the convention that scalar products are antilinear in the first variable and linear in the second. Let $\mathscr{H}$ denote the finite-dimensional Hilbert space obtained by considering $\mathscr{A}$ with scalar product (1.1). The algebra $\mathscr{A}$ acts on $\mathscr{H}$ by left multiplication and we denote by A the image of $\mathscr{A}$ in $\mathscr{L}(\mathscr{H})$ in this representation.

In what follows the algebras $\mathscr{A}$ and A will be identified and the symbols $\mathscr{A}$ and A will be used interchangeably depending on the context.

Consider the linear map

$$\mathscr{A} \otimes \mathscr{A} \ni (a \otimes b) \longmapsto \Delta(a)(\mathbb{1} \otimes b) \in \mathscr{A} \otimes \mathscr{A}. \tag{1.2}$$

It is well known that this map is invertible with inverse given by $a \otimes b \longmapsto \big((\mathrm{id} \otimes S)\Delta(a)\big)(\mathbb{1} \otimes b)$, where S is the antipode of $(\mathscr{A}, \Delta)$. Let W be the same map only considered as a mapping $\mathscr{H} \otimes \mathscr{H}$. The difference is almost negligible, but it makes sense to keep the distinction. The invariance of $\boldsymbol{h}$ immediately implies that $W \in \mathscr{L}(\mathscr{H} \otimes \mathscr{H})$ is unitary:

$$\begin{aligned}\langle W(a_1 \otimes b_1)|W(a_2 \otimes b_2)\rangle &= (\boldsymbol{h} \otimes \boldsymbol{h})\big((\mathbb{1} \otimes b_1^*)\Delta(a_1)^*\Delta(a_2)(\mathbb{1} \otimes b_2)\big)\\ &= (\boldsymbol{h} \otimes \boldsymbol{h})\big((\mathbb{1} \otimes b_1^*)\Delta(a_1^*a_2)(\mathbb{1} \otimes b_2)\big)\\ &= \boldsymbol{h}\big(b_1^*\big((\boldsymbol{h} \otimes \mathrm{id})\Delta(a_1^*a_2)\big)b_2\big)\\ &= \boldsymbol{h}(b_1^*b_2)\boldsymbol{h}(a_1^*a_2) = \langle a_1 \otimes b_1|a_2 \otimes b_2\rangle\end{aligned}$$

for all $a_1, a_2, b_1, b_2 \in \mathscr{H}$.

Given $a, b \in \mathscr{H}$ we define $\omega_{a,b}$ to be the linear functional on $\mathscr{L}(\mathscr{H})$ given by

$$T \longmapsto \langle a|Tb\rangle .$$

Such functionals span all of $\mathscr{L}(\mathscr{H})^*$. Using the fact that $\mathscr{L}(\mathscr{H} \otimes \mathscr{H}) \cong \mathscr{L}(\mathscr{H}) \otimes \mathscr{L}(\mathscr{H})$ in a canonical way, we can consider the *left slice* $(\omega_{a,b} \otimes \mathrm{id})W$ of W with $\omega_{a,b}$. It is easily checked that $(\omega_{a,b} \otimes \mathrm{id})W$ is the operator on $\mathscr{H}$ which acts as left multiplication by

$$(\boldsymbol{h} \otimes \mathrm{id})\big((a^* \otimes \mathbb{1})\Delta(b)\big) \in \mathscr{A}.$$

Noting that just like (1.2), the map $a \otimes b \mapsto (a \otimes \mathbb{1})\Delta(a)$ is surjective (Hopf $*$-algebras are regular, [6]) we find that

$$\big\{(\omega \otimes \mathrm{id})W|\omega \in \mathscr{L}(\mathscr{H})^*\big\} = \mathsf{A}.$$

Another important feature of W is that it is a *multiplicative unitary*, i.e. the *pentagon equation* holds: we have

$$W_{23}W_{12}W_{23}^* = W_{12}W_{13}. \tag{1.3}$$

The notation of (1.3) requires explanation: we define $W_{12} = W \otimes \mathbb{1}_{\mathscr{H}}$, $W_{23} = \mathbb{1}_{\mathscr{H}} \otimes W$ and $W_{13} = (\Sigma \otimes \mathbb{1}_{\mathscr{H}})W_{23}(\Sigma \otimes \mathbb{1}_{\mathscr{H}})$, where Σ denotes the flip map $\mathscr{H} \otimes \mathscr{H} \ni (a \otimes b) \mapsto (b \otimes a) \in \mathscr{H} \otimes \mathscr{H}$. This kind of notation is known as the *leg numbering notation*. In what follows we will use this notation with up to five tensor factors and not only for linear operators on tensor products, but also for elements of multiple tensor products of algebras. Multiplicative unitaries were introduced in [7] and the theory was later developed in many papers (e.g. [8–11]).

The comultiplication Δ is *implemented* by W in the following sense: given $a \in \mathsf{A} \cong \mathscr{A}$ the operator on $\mathscr{H} \otimes \mathscr{H}$ of left multiplication by $\Delta(a)$ is

$$W(a \otimes \mathbb{1})W^*.$$

Thus transporting Δ from $\mathscr{A}$ onto A we obtain a comultiplication on A which by (1.3) satisfies

$$(\mathrm{id} \otimes \Delta)W = W_{12}W_{13}.$$

Similarly, transporting the antipode S onto A we find that

$$(\mathrm{id} \otimes S)W = W^*. \tag{1.4}$$

Indeed, given $a, a', b, b' \in \mathscr{H}$ we can calculate

$$\begin{aligned}\big\langle a'|S\big((\omega_{a,b} \otimes \mathrm{id})W\big)b'\big\rangle &= \boldsymbol{h}\big(a'^*S((\boldsymbol{h} \otimes \mathrm{id})((a^* \otimes \mathbb{1})\Delta(b)))b'\big)\\ &= (\boldsymbol{h} \otimes \boldsymbol{h})\big((a^* \otimes a'^*)((\mathrm{id} \otimes S)\Delta(b))(\mathbb{1} \otimes b')\big)\,\big\langle a'|(\omega_{a,b} \otimes \mathrm{id})(W^*)b'\big\rangle.\end{aligned}$$

Let

$$\widehat{\mathsf{A}} = \big\{(\mathrm{id} \otimes \omega)W|\omega \in \mathscr{L}(\mathscr{H})^*\big\}.$$