Journal of Geometry and Physics 89 (2015) 38–49

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

# Almost isometries of non-reversible metrics with applications to stationary spacetimes

Miguel Angel Javaloyes [a,*], Leandro Lichtenfelz [b], Paolo Piccione [b]

[a] *Departamento de Matemáticas, Universidad de Murcia, Campus de Espinardo, 30100 Espinardo, Murcia, Spain*

[b] *Departamento de Matemática, Universidade de São Paulo, Rua do Matão 1010, CEP 05508-900, São Paulo, SP, Brazil*

ARTICLE INFO

*Article history:*
Received 13 January 2014
Accepted 2 December 2014
Available online 10 December 2014

*MSC:*
primary 53C22
53C50
53C60
58B20

*Keywords:*
Finsler and Randers metrics
Geodesics
Isometries
Quasi-metrics
Stationary spacetimes

ABSTRACT

We develop the basics of a theory of almost isometries for spaces endowed with a quasi-metric. The case of non-reversible Finsler (more specifically, Randers) metrics is of particular interest, and it is studied in more detail. The main motivation arises from General Relativity, and more specifically in spacetimes endowed with a timelike conformal field $K$, in which case *conformal diffeomorphisms* correspond to almost isometries of the Fermat metrics defined in the spatial part. A series of results on the topology and the Lie group structure of conformal maps are discussed.



## 1. Introduction

A quasi-metric is just a metric (in the context of metric spaces) without the restriction of symmetry. In the last years, an increasing interest in quasi-metrics has risen [1–8]. This interest can be justified by the amount of applications, since non-reversibility is present in many situations.

The first systematic study of quasi-metrics was carried out by W.A. Wilson in [9]. Later, also H. Busemann developed some results on metrics without the restriction of symmetry [10], but with the additional condition that forward and backward balls generate the same topology (see Remark 2.2). Most of the results by H. Busemann were developed for (symmetric) metric spaces and it was E.M. Zaustinsky who extended some of them to quasi-metric spaces [11].

Our main goal is to study the automorphisms of a quasi-metric space $(X, d)$ that preserve what we call the triangular function, namely, the quantity $T(x, y, z) = d(x, y) + d(y, z) - d(x, z)$. This quantity measures how far the points $x, y, z$ are from achieving the equality in the triangle inequality. Maps that preserve the triangular function will be called *almost isometries* and it is an immediate consequence of the very definition that such maps preserve minimizing geodesics (see Corollary 2.7). It is easy to see that when symmetry holds, almost isometries are in fact isometries. Therefore, this notion is relevant only in the non-symmetric context.

* Corresponding author.
*E-mail addresses:* majava@um.es (M.A. Javaloyes), silverratio@gmail.com (L. Lichtenfelz), piccione.p@gmail.com (P. Piccione).

http://dx.doi.org/10.1016/j.geomphys.2014.12.001

One of the fundamental examples of quasi-metrics is given by the distance associated to a non-reversible Finsler metric. As in the case of isometries, the almost isometries can be described in terms of the pullback of the Finsler metric. More precisely, they are the maps that preserve the Finsler metric up to an exact one-form, namely, the pullback of a Finsler metric is the sum of the same Finsler metric and the differential of a smooth function. One of the simplest examples of non-reversible Finsler metrics are Randers metrics, which are defined as the sum of the square root of a Riemannian metric and a one-form having norm less than one at every point. Recently, a relation between Randers metrics and standard stationary spacetimes has been explored [12–21]. The core of this relation lies in the fact that, as a consequence of the Fermat principle, lightlike geodesics project up to parametrization into geodesics of a Randers metric that we call Fermat metric. Then, studying existence and multiplicity of lightlike geodesics between an event and a vertical line is equivalent to studying existence and multiplicity of geodesics of the Fermat metric [12,14–16]. Moreover, causality of the stationary spacetime can be characterized in terms of completeness properties of the Fermat metric [17] and its causal boundary can be described in terms of the topological boundary of the Fermat metric [19]. Giving continuity to the fruitful interplay between Randers metrics and stationary spacetimes, we will explore this relation in the level of transformation groups. In fact, we will relate the conformal maps of the conformastationary spacetime that preserve the timelike Killing vector field with the almost isometries of the Fermat metric obtaining, as a consequence of this interplay, results of genericity and compactness for the $K$-conformal group. This relation was the departing point and the inspiration to get to the concept of almost isometry.

Let us describe in detail the results of this paper. In Section 2, we introduce the notions of triangular function of a quasi-metric and almost isometry (Definition 2.6), the latter being a map that preserves the triangular function. Then we show in Proposition 2.8 that the definition of almost isometry $\varphi$ of $(X, d)$ is equivalent to the existence of a real function $f : X \to \mathbb{R}$ such that

$$d(\varphi(x), \varphi(y)) = d(x, y) + f\big(\varphi(y)\big) - f\big(\varphi(x)\big)$$

for every $x, y \in X$. Moreover, we show that $\varphi$ is always a homeomorphism and $f$ is continuous (Lemma 2.10). Finally we show that the extended isometry group $\widetilde{\mathrm{Iso}}(X, d)$, made up by the almost isometries of $(X, d)$, is contained in the isometry group of the symmetrized metric in (1) and it is a topological group (Proposition 2.11). In Section 2.1, we study local almost isometries and we conclude in Theorem 2.14 that a local almost isometry between two length spaces with weakly finitely compact domain and simply connected codomain must be a global almost isometry. A counterexample to this result in the case that the quasi-metric spaces are not length spaces is provided in Remark 2.15.

In Section 3 we prove that an almost isometry of a Finsler manifold is an isometry of the symmetrized Finsler metric in (7) and then it is smooth (Lemma 3.1). Moreover, the function $f : M \to \mathbb{R}$ of Proposition 2.8 is smooth and $\varphi$ is an almost isometry for $F$ if and only if $\varphi_*(F) = F - \mathrm{d}f$ (Proposition 3.2). Finally, the extended isometry group of $(S, F)$, denoted by $\widetilde{\mathrm{Iso}}(S, F)$, is a closed subgroup of the isometry group of the symmetrized Finsler metric $\hat{F}$, which is a Lie group (see for instance [22]) and then $\widetilde{\mathrm{Iso}}(S, F)$ is also a Lie group (Proposition 3.3).

In Section 4 we first introduce $K$-conformastationary decompositions $(M = S \times \mathbb{R}, g^K)$ of conformastationary spacetimes endowed with a complete timelike conformal field $K$ in (13) and the Fermat metric $F^K$ associated to them (14). In Theorem 4.3 we show that a conformal map $\psi : (M, g) \to (M, g)$ determines an almost isometry $\varphi : (S, F^K) \to (S, F^W)$ of the Fermat metrics $F^K$ and $F^W$ associated to one of the conformastationary decompositions determined respectively by $K$ and $W = \psi_*(K)$. In particular $\varphi$ is given by the spatial component of the conformal map $\psi$. Then we define a $K$-conformal map as a map that is conformal and preserve the timelike conformal vector field $K$. In Proposition 4.7 we show that an almost isometry determines a $K$-conformal map up to a composition with an element of the closed subgroup generated by $K$, which will be denoted by $\mathcal{K}$. Indeed, there is a Lie group homomorphism between the $K$-conformal maps quotiented by $\mathcal{K}$ and the extended isometry group. Furthermore, in Corollaries 4.8 and 4.10, we use the former Lie group homomorphism to obtain a genericity result for standard stationary spacetimes with discrete $K$-conformal group and the compactness of the $K$-conformal group.

In the last subsection of Section 4, we obtain some consequences for the conformal group $\mathrm{Conf}(M, g)$. In particular in Corollary 4.13 we give a characterization of the compactness of $\mathrm{Conf}(M, g)/\mathcal{K}$. Finally in Theorem 4.14 we collect the one-to-one relation between conformal maps and almost isometries of the Fermat metrics up to composition with elements of $\mathcal{K}$.

## 2. Quasi-metrics and almost isometries

Let us first of all introduce the concept of a quasi-metric (see [9]).

**Definition 2.1.** Given a set $X$, we say that a function $d : X \times X \to \mathbb{R}$ is a *quasi-metric* if

(i) $d(x, y) \geq 0$ for every $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) + d(y, z) \geq d(x, z)$ (triangle inequality).

As a consequence of the lack of symmetry, there are two kinds of balls, namely, *forward and backward balls*, defined by $B^+_d(x, r) = \{y \in X : d(x, y) < r\}$ and $B^-_d(x, r) = \{y \in X : d(y, x) < r\}$ respectively, for $x \in X$ and $r > 0$. Both families generate two topologies that we will call respectively forward and backward topologies.