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The local geometry of compact homogeneous Lorentz spaces

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ABSTRACT

In 1995, S. Adams and G. Stuck as well as A. Zeghib independently provided a classification of non-compact Lie groups which can act isometrically and locally effectively on compact Lorentzian manifolds. In the case that the corresponding Lie algebra contains a direct summand isomorphic to the two-dimensional special linear algebra or to a twisted Heisenberg algebra, Zeghib also described the geometric structure of the manifolds. Using these results, we investigate the local geometry of compact homogeneous Lorentz spaces whose isometry groups have non-compact connected components. It turns out that they all are reductive. We investigate the isotropy representation and curvatures. In particular, we obtain that any Ricci-flat compact homogeneous Lorentz space is flat or has compact isometry group.

1. Introduction

The aim of our work is a detailed investigation of compact homogeneous Lorentzian manifolds whose isometry groups have non-compact connected components. We will provide the most important results and proofs of [1]. Focusing on the concepts and the main ideas, we refer the reader to [1] for more details.

In Section 2, we introduce the notations and theorems concerning isometric and (locally) effective actions of Lie groups on compact Lorentzian manifolds. Let us shortly describe the content of the basic theorems, Theorems 1–4. Adams and Stuck [2] as well as Zeghib [3] independently provided an algebraic classification of Lie groups that act isometrically and locally effectively on Lorentzian manifolds that are compact. More generally, Zeghib showed the same result for manifolds of finite volume. We will shortly describe the approach of Zeghib for the proof of Theorem 2. Theorem 2 states that if a Lie group G is acting isometrically and locally effectively on a Lorentzian manifold $M = (M, g)$ of finite volume, then there exist Lie algebras $\mathfrak{k}$, $\mathfrak{a}$, and $\mathfrak{s}$ such that the Lie algebra $\mathfrak{g}$ of G is equal to the direct sum $\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{s}$. Here, $\mathfrak{k}$ is compact semisimple, $\mathfrak{a}$ is abelian, and $\mathfrak{s}$ is isomorphic to one of the following:

- the trivial algebra,
- the two-dimensional affine algebra $\mathfrak{aff}(\mathbb{R})$,
- the $(2d+1)$-dimensional Heisenberg algebra $\mathfrak{he}_d$,
- a certain $(2d+2)$-dimensional twisted Heisenberg algebra $\mathfrak{he}_d^\lambda$ $(\lambda \in \mathbb{Z}_+^d)$,
- the two-dimensional special linear algebra $\mathfrak{sl}_2(\mathbb{R})$.

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The key idea behind the proof is introducing a certain symmetric bilinear form κ on $\mathfrak{isom}(M)$, the Lie algebra of the isometry group of (M, g). For $X, Y \in \mathfrak{isom}(M)$,

$$\kappa(X, Y) := \int_M g(\widetilde{X}, \widetilde{Y})(x)d\mu(x).$$

Here, $\widetilde{X}, \widetilde{Y}$ are complete Killing vector fields corresponding to X, Y. In the case that M is not compact but has finite volume, one has to restrict the integration to an $\mathrm{Isom}(M)$-invariant non-empty open subset of M such that $|g(\widetilde{X}, \widetilde{Y})|$ is bounded by a constant depending only on X and Y. Such an open set always exists. κ induces in a canonical way a symmetric bilinear form on the Lie algebra $\mathfrak{g}$ of a Lie group G acting isometrically and locally effectively on M.

κ is ad-invariant and fulfills the so-called condition $(\star)$. A symmetric bilinear form b on the Lie algebra $\mathfrak{g}$ of a Lie group G fulfills this condition if for any subspace V of $\mathfrak{g}$ such that the set of $X \in V$ generating a non-precompact one-parameter group in G is dense in V, the restriction of b to $V \times V$ is positive semidefinite, and its kernel has dimension at most one. Proposition 2.2 shows that κ fulfills this condition.

Condition $(\star)$ is the main tool for proving Theorem 1. Theorem 1 describes the algebraic structure of connected non-compact Lie groups G whose Lie algebras $\mathfrak{g}$ possess an ad-invariant symmetric bilinear form κ fulfilling condition $(\star)$. The structure of the Lie algebras is exactly the same as the one in Theorem 2. Furthermore, in the latter two cases, if G is contained in the isometry group of the manifold, the subgroup generated by $\mathfrak{s}$ has compact center if and only if the subgroup is closed in the isometry group. This formulation corrects the claim of [3] that the subgroup generated by $\mathfrak{s}$ has compact center.

Theorem 3, which was stated and shown in its general form in [2], says that the subgroup generated by $\mathfrak{s}$ in Theorem 2 acts locally freely on M. This result is important for the geometric characterization of compact Lorentzian manifolds in Theorem 4 (which is due to Zeghib). This theorem considers the case that the Lorentzian manifold M is compact and $\mathfrak{s}$ is isomorphic to the two-dimensional special linear algebra or to a twisted Heisenberg algebra. If $\mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{R})$, M is covered isometrically by a warped product of the universal cover of the two-dimensional special linear group and a Riemannian manifold N. Else, if $\mathfrak{s} \cong \mathfrak{he}_d^\lambda$, M is covered isometrically by a twisted product $S \times_{Z(S)} N$ of a twisted Heisenberg group S and a Riemannian manifold N. Due to the correction of Theorem 1 compared to [3] and due to the fact that the invariance of a Lorentzian scalar product on a twisted Heisenberg algebra under the adjoint action of the nilradical (what Zeghib called *essential ad-invariance*) is equivalent to ad-invariance (see Proposition 2.4 (iii)), the formulation of Theorem 4 is different from the one in [3].

Our main theorems are formulated in Section 3. We will show them in Section 4, where we provide an analysis of compact homogeneous Lorentz spaces M, in particular of those whose isometry groups have non-compact connected components. We start by presenting a topological and geometric description of them in Theorem 5, which is also slightly different from the corresponding result of Zeghib. Essentially, it shows that if the isometry group of M has non-compact connected components, M is covered isometrically either by a metric product of the universal cover of the two-dimensional special linear group and a compact homogeneous Riemannian manifold N, or by a twisted product $S \times_{Z(S)} N$ of a twisted Heisenberg group S and a compact homogeneous Riemannian manifold N. Additionally, we give a covering of the identity component of $\mathrm{Isom}(M)$, which will be important for the investigation of the local geometry of M.

Theorem 6 shows what was originally stated in [3], namely that the connected components of the isotropy group are compact. We give an elegant proof using the ideas of Adams and Stuck in the proof of Theorem 3. Moreover, it turns out that every compact homogeneous Lorentzian manifold has a reductive representation defined in a natural way. For this representation, the induced bilinear form κ plays an essential role.

Using a slightly different reductive representation than in Theorem 6 in the case that the isometry group contains a twisted Heisenberg group as a subgroup, we are able to describe the local geometry of compact homogeneous Lorentzian manifolds whose isometry groups have non-compact connected components in terms of the curvature of the manifold. We also investigate the isotropy representation of the manifold, our result concerning a decomposition into (weakly) irreducible summands being summarized in Theorem 7.

Our results of Section 4.4 directly yield the proof of Theorem 8 which states that the isometry group of any Ricci-flat compact homogeneous Lorentzian manifold has compact connected components. Together with two results in [4,5], it follows that the isometry group of any Ricci-flat compact homogeneous Lorentzian manifold that is non-flat is in fact compact. Note that certain Cahen–Wallach spaces, which all are symmetric Lorentzian spaces, are Ricci-flat and non-flat. It is not known whether at least all Ricci-flat compact homogeneous Lorentzian manifolds are flat.

2. Lie groups acting isometrically on compact Lorentzian manifolds

This section is devoted to describe the classification of Lie algebras of Lie groups acting isometrically and locally effectively on compact Lorentzian manifolds. Following the approach of Zeghib in [3], a certain symmetric bilinear form κ on Lie algebras plays an important role. We also state a theorem about local freeness of the action of a subgroup of the isometry group, which is due to [2]. Following [3], we finally describe the topological and geometric structure of the manifolds.