Journal of Geometry and Physics 89 (2015) 82–110

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Symplectic reduction of holonomic open-chain multi-body systems with constant momentum

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ARTICLE INFO

Article history:
Received 10 March 2014
Accepted 16 December 2014
Available online 24 December 2014

Keywords:
Open-chain multi-body system
Differential forms
Hamiltonian dynamics
Momentum map
Symplectic reduction

ABSTRACT

This paper presents a two-step symplectic geometric approach to the reduction of Hamilton's equation for open-chain, multi-body systems with multi-degree-of-freedom holonomic joints and constant momentum. First, symplectic reduction theorem is revisited for Hamiltonian systems on cotangent bundles. Then, we recall the notion of displacement subgroups, which is the class of multi-degree-of-freedom joints considered in this paper. We briefly study the kinematics of open-chain multi-body systems consisting of such joints. And, we show that the relative configuration manifold corresponding to the first joint is indeed a symmetry group for an open-chain multi-body system with multi-degree-of-freedom holonomic joints. Subsequently using symplectic reduction theorem at a non-zero momentum, we express Hamilton's equation of such a system in the symplectic reduced manifold, which is identified by the cotangent bundle of a quotient manifold. The kinetic energy metric of multi-body systems is further studied, and some sufficient conditions are introduced, under which the kinetic energy metric is invariant under the action of a subgroup of the configuration manifold. As a result, the symplectic reduction procedure for open-chain, multi-body systems is extended to a two-step reduction process for the dynamical equations of such systems. Finally, we explicitly derive the reduced dynamical equations in the local coordinates for an example of a six-degree-of-freedom manipulator mounted on a spacecraft, to demonstrate the results of this paper.

1. Introduction

In order to better understand the behaviour of Hamiltonian and Lagrangian systems, researchers have been trying to find *conserved quantities* that are used to integrate a part of dynamical equations, and derive closed-form equations for some parameters of such systems. For example, Jacobi in 1884 introduced Hamilton–Jacobi equations, which give the necessary conditions for integrability of a Lagrangian system [1]. Also, Emmy Noether in 1918 in her famous paper [2] proved that any symmetry of the action functional of a Lagrangian system corresponds to a conserved quantity. This result is an inflection point in identifying conserved quantities, and its relation with the reduction of dynamical equations of a system. By *reducing the dynamical equations* we mean expressing the differential equations representing a (Lagrangian or Hamiltonian) system on a manifold whose dimension is less than the original phase space of the system, by quotienting a group action and

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http://dx.doi.org/10.1016/j.geomphys.2014.12.011
0393-0440/

Operators

L_r	Left composition/translation by r
R_r	Right composition/translation by r
K_r	Conjugation by r
Ad_r	Adjoint operator corresponding to r
ad_ξ	adjoint operator corresponding to ξ
$[\xi, \eta]$	Lie bracket or matrix commutator
$T_m f$	Tangent map corresponding to the map f at the element m
$T_m^* f$	Cotangent map corresponding to the map f at the element m
$T_m M$	Tangent space of the manifold M at the element m
TM	Tangent bundle of the manifold M
$T_m^* M$	Cotangent space of the manifold M at the element m
$T^* M$	Cotangent bundle of the manifold M
$\exp(\xi)$	Group/matrix exponential of ξ
$Lie(G)$	Lie algebra of the Lie group G
$Lie^*(G)$	Dual of the Lie algebra of the Lie group G
G_μ	Coadjoint isotropy group for $\mu \in Lie^*(G)$
$\ltimes$	Semi-direct product of groups
$\ll \cdot, \cdot \gg$	Euclidean metric
$\|v\|_h$	Norm of the vector v with respect to the metric h
$\langle \cdot, \cdot \rangle$	Canonical pairing of the elements of tangent and cotangent space
$\mathcal{L}_X$	Lie derivative with respect to the vector field X
ξ_M	Vector field on the manifold M induced by the infinitesimal action of $\xi \in Lie(G)$
$\iota_X \Omega$	Interior product of the differential form Ω by the vector field X
$\mathfrak{X}(M)$	Space of all vector fields on the manifold M
$\Omega^2(M)$	Space of all differential 2-forms on the manifold M
$d\Omega$	Exterior derivative of the differential form Ω
dH	Exterior derivative of the function H
M/G	Quotient manifold corresponding to a free and proper action of the Lie group G

eliminating the trivial behaviour of the system or restricting the system to a submanifold of the phase space. In the following, we first review two existing reduction theories for Hamiltonian and Lagrangian mechanical systems. Then, we report the reduction methods for multi-body systems, and finally, we state the contributions of this paper.

1.1. Background

1.1.1. Reduction theories

From the geometric point of view, a Hamiltonian system is a vector field X on a symplectic manifold (M, Ω) (phase space) that satisfies (coordinate-independent) Hamilton's equation

$$\iota_X \Omega = dH,$$

where $\iota_X \Omega$ is the interior product of the vector field X with the symplectic form Ω, and the function $H : M \to \mathbb{R}$ is the Hamiltonian of the system. In this formulation, if H and Ω are invariant under a group action, then there exists a conserved quantity (*momentum*) for the Hamiltonian system and we can reduce Hamilton's equation [3]. In this reduction process, we have to take care of not only the topology of the phase space and its symplectic structure, but also the Hamiltonian H and its corresponding Hamiltonian vector field X. As for the reduction of the phase space along with its symplectic structure (M, Ω), the symplectic reduction theorem by Marsden and Weinstein [4] gives an instruction to find the reduced phase space and its symplectic structure. In the following, we state this theorem, and report its impact on the geometric mechanics literature.

Let G be a Lie group, and M be the phase space of a system. The symplectic reduction theorem states that in the presence of a free and proper G-action and an (Ad^*-equivariant) momentum map $\mathbf{M} : M \to Lie^*(G)$, for any value $\mu \in Lie^*(G)$ of the momentum the quotient manifold $M_\mu := \mathbf{M}^{-1}(\mu)/G_\mu$ inherits a symplectic form Ω_μ. Here, G_μ is the coadjoint isotropy group of μ, Ω_μ is identified by the equality $i_\mu^* \Omega = \pi_\mu^* \Omega_\mu$, and the maps $i_\mu : \mathbf{M}^{-1}(\mu) \hookrightarrow M$ and $\pi_\mu : \mathbf{M}^{-1}(\mu) \to \mathbf{M}^{-1}(\mu)/G_\mu$ are the canonical inclusion and projection maps [4]. The pair (M_μ, Ω_μ) is called the *symplectic reduced manifold*. This theorem by Marsden and Weinstein made a huge impact on unifying the reduction methods that had been previously developed for Lagrangian and Hamiltonian systems, such as classical Routh method and the reduction of Lagrangian systems by cyclic parameters [5].

For a *mechanical system*, the phase space is the cotangent bundle of the configuration manifold $T^*\mathcal{Q}$ that admits a canonical symplectic 2-form, which is $\Omega_{can} := -dp \wedge dq$, in coordinates. As the result, $(T^*\mathcal{Q}, \Omega_{can})$ is a symplectic manifold.