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Connes' calculus for the quantum double suspension

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ABSTRACT

Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ Connes associated a canonical differential graded algebra $\Omega_D^\bullet(\mathcal{A})$. However, so far this has been computed for very few special cases. We identify suitable hypotheses on a spectral triple that helps one to compute the associated Connes' calculus for its quantum double suspension. This allows one to compute $\Omega_D^\bullet$ for spectral triples obtained by iterated quantum double suspension of the spectral triple associated with a first order differential operator on a compact smooth manifold. This gives the first systematic computation of Connes' calculus for a large family of spectral triples.

1. Introduction

Noncommutative geometry of Connes is the study of interplay between an algebra $\mathcal{A}$ and a selfadjoint operator D, often referred as the Dirac operator represented on the same Hilbert space $\mathcal{H}$. In the last chapter of his book [1], using these ingredients Connes also constructed a calculus $\Omega_D^\bullet(\mathcal{A})$. Please recall that by a calculus or a differential calculus one often means a differential graded algebra. It is shown in [1] that using this calculus one can produce Hochschild cocycle and cyclic cocycle (under certain assumptions) for Poincaré dual algebras which makes $\Omega_D^\bullet$ worth studying. Last but not the least Connes showed that in the case of classical spectral triple associated to a compact Riemannian spin manifold, $\Omega_D^\bullet$ gives back the space of de-Rham forms on the manifold. This establishes $\Omega_D^\bullet$ as a genuine noncommutative generalization of the classical de-Rham complex of a manifold. There are other instances of calculus as well, see for example [2–4]. For a better understanding of the Connes calculus it is imperative that we compute this in some cases. However outside the works of Connes there are very few instances [5,6] where this calculus have been computed and used for further investigation [7]. In view of this scenario, here in this article we set ourselves with the task of computation of the Connes calculus for a certain systematic class of examples, which is by far missing till date.

The concept of quantum double suspension (QDS) of an algebra $\mathcal{A}$, denoted by $\Sigma^2\mathcal{A}$, was introduced by Hong–Szymański in [8]. Later quantum double suspension of a spectral triple was introduced by Chakraborty–Sundar [9] and a class of examples of regular spectral triple having simple dimension spectrum were constructed, useful in the context of local index formula of Connes–Moscovici [10]. Note that iterating QDS on a manifold one can produce genuine noncommutative spectral triples. Under the following hypotheses

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- $[D, a]F - F[D, a]$ is a compact operator for all $a \in \mathcal{A}$, where F is the sign of the operator D,
- $\mathcal{H}^\infty := \bigcap_{k\geq 1} \mathcal{D}om(D^k)$ is a left $\mathcal{A}$-module and $[D, \mathcal{A}] \subseteq \mathcal{A} \otimes \mathcal{E}nd_{\mathcal{A}}(\mathcal{H}^\infty) \subseteq \mathcal{E}nd_{\mathbb{C}}(\mathcal{H}^\infty)$,

we compute $\Omega_D^\bullet$ for the quantum double suspended spectral triple $(\Sigma^2\mathcal{A}, \Sigma^2\mathcal{H}, \Sigma^2 D)$. Notable features of these hypotheses are, firstly the spectral triple associated with a first order elliptic differential operator on a manifold will always satisfy them and secondly they are stable under quantum double suspension. Thus our results allows one to compute the Connes calculus for spectral triples obtained by iteratively quantum double suspending spectral triples associated with first order differential operators on smooth compact manifolds. In particular iterated application of our construction on the spectral triple $\left(C^\infty(\mathbb{T}), L^2(\mathbb{T}), \frac{1}{i}\frac{d}{d\theta}\right)$ imply the computation of the Connes calculus for odd dimensional quantum spheres. This extends earlier work of [5].

Organization of this paper is as follows. In Section 2 we go through the definition of Connes' calculus $\Omega_D^\bullet$ and the quantum double suspension. Section 3 is devoted to the computation of the space of forms $\Omega^\bullet_{\Sigma^2 D}(\Sigma^2\mathcal{A})$. In the last section we study compatible connections, curvatures on a *Hermitian* finitely generated projective (right)module [1, Chapter 6] over $\Sigma^2\mathcal{A}$. If we take $\mathcal{E}$ to be a Hermitian finitely generated projective module over $\mathcal{A}$, and denote the affine space of compatible connections by $Con(\mathcal{E})$, then there is a canonical Hermitian finitely generated projective module over $\Sigma^2\mathcal{A}$ which we denote by $\widetilde{\mathcal{E}}$. The affine space of compatible connections on $\widetilde{\mathcal{E}}$ is denoted by $Con(\widetilde{\mathcal{E}})$. We show that there is an affine embedding of $Con(\mathcal{E})$ into $Con(\widetilde{\mathcal{E}})$ which preserves the Grassmannian connection and together with $Hom_{\mathcal{A}}\left(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^2(\mathcal{A})\right)$, the vector space containing the subspace of curvatures, these fit into a commutative diagram.

2. Preliminaries on Connes' calculus and the quantum double suspension

In this section we recall the definition of Connes' calculus $\Omega_D^\bullet$ from [1] and that of the quantum double suspension from [8,9].

Definition 2.1. A *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ over an associative algebra $\mathcal{A}$ with involution $\star$ consists of the following things:

1. a $\star$-representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$,
2. an unbounded selfadjoint operator D,
3. D has compact resolvent and $[D, a]$ extends to a bounded operator on $\mathcal{H}$ for every $a \in \mathcal{A}$.

We shall assume that $\mathcal{A}$ is unital and the unit $1 \in \mathcal{A}$ acts as the identity on $\mathcal{H}$. If $|D|^{-p}$ is in the ideal of Dixmier traceable operators $\mathcal{L}^{(1,\infty)}$ then we say that the spectral triple is p-summable. Moreover, if there is a $\mathbb{Z}_2$-grading $\gamma \in \mathcal{B}(\mathcal{H})$ such that γ commutes with every element of $\mathcal{A}$ and anticommutes with D, then the spectral triple $(\mathcal{A}, \mathcal{H}, D, \gamma)$ is said to be an *even spectral triple*.

Definition 2.2. Let $\Omega^\bullet(\mathcal{A}) = \bigoplus_{k=0}^\infty \Omega^k(\mathcal{A})$ be the reduced universal differential graded algebra over $\mathcal{A}$. Here $\Omega^k(\mathcal{A}) := \mathcal{A} \otimes \bar{\mathcal{A}}^{\otimes k}$, $\bar{\mathcal{A}} = \mathcal{A}/\mathbb{C}$. The graded product is given by

$$\left(\sum_k a_{0k} \otimes \overline{a_{1k}} \otimes \cdots \otimes \overline{a_{mk}}\right) \cdot \left(\sum_{k'} b_{0k'} \otimes \overline{b_{1k'}} \otimes \cdots \otimes \overline{b_{nk'}}\right) := \sum_{k,k'} a_{0k} \otimes (\otimes_{j=1}^{m-1}\overline{a_{jk}}) \otimes \overline{a_{mk}b_{0k'}} \otimes (\otimes_{i=1}^{n}\overline{b_{ik'}})$$
$$+ \sum_{i=1}^{m-1}(-1)^i a_{0k} \otimes \overline{a_{1k}} \otimes \cdots \otimes \overline{a_{m-i,k}a_{m-i+1,k}} \otimes \cdots \otimes \overline{a_{mk}} \otimes (\otimes_{i=0}^{n}\overline{b_{ik'}}) + (-1)^m a_{0k}a_{1k} \otimes (\otimes_{j=2}^{m}\overline{a_{jk}}) \otimes (\otimes_{i=0}^{n}\overline{b_{ik'}})$$

for $\sum_k a_{0k} \otimes \overline{a_{1k}} \otimes \cdots \otimes \overline{a_{mk}} \in \Omega^m(\mathcal{A})$ and $\sum_{k'} b_{0k'} \otimes \overline{b_{1k'}} \otimes \cdots \otimes \overline{b_{nk'}} \in \Omega^n(\mathcal{A})$. There is a differential d acting on $\Omega^\bullet(\mathcal{A})$, given by

$$d(a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_k}) := 1 \otimes \overline{a_0} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_k} \quad \forall a_j \in \mathcal{A},$$

and it satisfies the relations

1. $d^2\omega = 0$, $\forall\, \omega \in \Omega^\bullet(\mathcal{A})$,
2. $d(\omega_1\omega_2) = (d\omega_1)\omega_2 + (-1)^{deg(\omega_1)}\omega_1 d\omega_2$, $\forall\, \omega_j \in \Omega^\bullet(\mathcal{A})$.

We have a $*$-representation π of $\Omega^\bullet(\mathcal{A})$ on $\mathcal{Q}(\mathcal{H}) := \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, given by

$$\pi(a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_k}) := a_0[D, a_1] \ldots [D, a_k] + \mathcal{K}(\mathcal{H}); \quad a_j \in \mathcal{A}.$$

Let $J_0^{(k)} = \{\omega \in \Omega^k : \pi^k(\omega) = 0\}$ and $J' = \bigoplus J_0^{(k)}$. But J' fails to be a differential ideal in $\Omega^\bullet$. We consider $J^\bullet = \bigoplus J^{(k)}$ where $J^{(k)} = J_0^{(k)} + dJ_0^{(k-1)}$. Then $J^\bullet$ becomes a differential graded two-sided ideal in $\Omega^\bullet$ and hence, the quotient $\Omega_D^\bullet = \Omega^\bullet/J^\bullet$ becomes a differential graded algebra, called the Connes' calculus.

The representation π gives an isomorphism,

$$\Omega_D^k \cong \pi(\Omega^k)/\pi(dJ_0^{k-1}), \quad \forall\, k \geq 0. \tag{2.1}$$