Journal of Geometry and Physics 88 (2015) 94–104

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

A Lie algebra structure on variation vector fields along curves in 2-dimensional space forms

José del Amor [a], Ángel Giménez [b,*], Pascual Lucas [a]

[a] Departamento de Matemáticas, Universidad de Murcia, Campus del Espinardo, 30100 Murcia, Spain

[b] Centro de Investigación Operativa, Universidad Miguel Hernández de Elche, Avda, Universidad s/n, 03202 Elche (Alicante), Spain

ARTICLE INFO

Article history:
Received 5 April 2014
Received in revised form 19 September 2014
Accepted 18 November 2014
Available online 26 November 2014

MSC:
14H70
17B80
35Q53
37K05
37K10

Keywords:
Lie algebra
Integrable Hamiltonian system
Curve motions
Planar filament equation
mKdV equation

ABSTRACT

A Lie algebra structure on variation vector fields along an immersed curve in a 2-dimensional real space form is investigated. This Lie algebra particularized to plane curves is the cornerstone in order to define a Hamiltonian structure for plane curve motions. The Hamiltonian form and the integrability of the planar filament equation are finally discussed from this point of view.

1. Introduction

This paper is motivated by the investigation of relationships between geometric motions of curves in a certain space and Hamiltonian systems of PDE's. Hamiltonian systems have been and remain an active topic of research, as witnessed by a number of relevant publications in the last decades. There is a great number of papers dealing with the study of integrable systems associated with the evolution of plane (or n-dimensional) curves, specifically, evolution equations for the differential invariants of the curve, for example, its curvatures. The pioneering work [1] by Hasimoto about the motion of a vortex filament in a fluid and its connection with the cubic nonlinear Schrödinger (NLS) equation through the Hasimoto transformation was the first to suggest such a link. The *vortex filament flow* is the evolution equation for space curves modelling the motion of a one-dimensional vortex filament in an incompressible fluid, and is given by

$$\gamma_t = kB, \tag{1}$$

where $\gamma_t(s) = \gamma(s, t)$ denotes the evolving curve, parametrized by the arc-length s, and k is its curvature.

* Corresponding author. Tel.: +34 966658542.
E-mail address: a.gimenez@umh.es (Á. Giménez).

http://dx.doi.org/10.1016/j.geomphys.2014.11.010
0393-0440/

Based on this research, many other authors raised the issue of what other curves flows induce a completely integrable PDE for their curvatures. In [2] Lamb gave a general procedure that helps to identify a certain space curve evolution with a given integrable equation. Later, in [3–8] and other references therein, the authors provide Hamiltonian structures to the intrinsic geometry of curves in n-dimensional Riemannian manifolds from different backgrounds. Chou and Qu (see [9,10]) study motions of plane curves when the velocity vector fields are given in terms of differential invariants under other background geometries, considering Klein geometries instead of Euclidean ones. They have obtained, among other nonlinear equations, that KdV, Harry-Dym and Sawada–Kotera equations are induced by this kind of motions. Nevertheless, the common thread of these papers is the study of the relationship between differential invariants and Hamiltonian structures of PDE's.

However, there are very few papers addressing the integrability of the curve motion equation itself. In this regard, Langer and Perline (see for instance [11]) interpreted the Hasimoto transformation as a Poisson map transforming LIE to NLS, which allows to lift Poisson structures of NLS equation to appropriate Poisson structures on the space of curves, finding in this way an infinite hierarchy of local commuting generalized symmetries and conserved quantities in involution for the LIE equation as the pull-back of those for the NLS equation. Yasui and Sasaki (see [12]), based on [11], introduce a differential calculus on the space of asymptotically linear curves in order to clarify the integrability of the vortex filament equation by developing a recursion operator, symmetries and constants of motion. The *planar filament (PF)* equation [13,14] is given by

$$\gamma_t = \frac{1}{2}k^2 T + k' N. \tag{2}$$

In [13], the authors show that all the odd vector fields in the infinite sequence of commuting vector fields for LIE hierarchy are planarity-preserving, and this infinite sequence, starting with PF itself, is an integrable system. The PF flow induces the mKdV equation $k_t = k''' + \frac{3}{2}k^2 k'$, a well-known completely integrable PDE.

Drawing inspiration from the ideas of these papers, our aim here is to define a Lie algebra structure on the whole set of local variation vector fields along an immersed curve in a 2-dimensional space form. It should be remarked that the subspace consisting of local arc-length preserving variation vector fields is actually closed under bracket (Theorem 8). By using this Lie algebra particularized to plane curves we define a Lie algebra structure on the phase space of PF equation, i.e. on the arc-length preserving variation vector fields on planar curves (Theorem 11), that enables us to set a Hamiltonian structure as well as showing its integrability. To achieve this, we first construct a Lie algebras homomorphism between the Lie algebra of local arc-length variation vector fields and the Lie algebra of derivation vector fields. Then, we pull-back the standard Hamiltonian structure at the level of the curvature flow (the one for mKdV flow) to a Hamiltonian structure at the level of the curve flow (the one for PF equation) by means of such homomorphism. Our approach may be somewhat a formalization of the idea given in [15] in which a curve motion flow is defined to be integrable if it induces a completely integrable system of PDE for its curvatures. Our background will be the algebraic general formalism by Gelfand and Dickey (see [16] for more details) about Hamiltonian structures, even though we could have used another different algebraic framework for our purposes.

Lastly, let us add that it was shown in [17] that the PF equation can be interpreted physically as localized induction equation for boundary of vortex patch for ideal fluid flow in two dimensions. But, the PF equation has appeared in many other contexts, for instance, in [18] the authors present the results of experimental and theoretical study of two-dimensional vortex filament tangling, appearing in laser–matter interactions on nanosecond time scale. In two dimensions, the authors use planar filament equation as an evolution equation on plane curves.

Below we summarize the general outline of the rest of the paper. In Section 2 we introduce some definitions and properties about derivations, and afterwards we give the suitable mathematical framework to study our Hamiltonian structures. Then we move in Section 3 to study the space of immersed arc-length parametrized curves in a 2-dimensional real space form. In particular, we describe the space consisting of variation vector fields locally preserving arc-length parameter and derive important results for later use. Section 4 is the main section and is devoted to study the Lie algebra structure on local variation vector fields. In Section 5 we use such a Lie algebra to construct the Hamiltonian operator and give the Hamiltonian structure for planar filament equation, discussing also its integrability.

2. Preliminaries

In this section we present the algebraic general framework underlying in both finite and infinite dimensional Hamiltonian systems (see [16]). Firstly, we introduce the differential calculus which will be needed later.

2.1. Differential algebra $\mathcal{P}$

Let n be a positive integer and consider $u_0, u_1, \dots, u_{n-2}$ differentiable functions in the real variable x. Set

$$u_i^{(m)} = \frac{d^m u_i}{dx^m}, \quad \text{for } m \in \mathbb{N},\ i \in \{0, \dots, n-2\}.$$

Let $\mathcal{P}$ be the algebra of *local functions*, i.e. $\mathcal{P} = \bigcup_{j=1}^{\infty} \mathcal{P}_j$, where $\mathcal{P}_j$ is the algebra of locally analytic functions of $u_0, \dots, u_{n-2}$ and their derivatives up to order j (see [19–22]). In this algebra we define a derivation ∂ obeying

$$\begin{cases} \partial(fg) = (\partial f)g + f(\partial g), \\ \partial(u_i^{(m)}) = u_i^{(m+1)}, \end{cases}$$