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On affine maps on non-compact convex sets and some characterizations of finite-dimensional solid ellipsoids

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ABSTRACT

Convex geometry has recently attracted great attention as a framework to formulate general probabilistic theories. In this framework, convex sets and affine maps represent the state spaces of physical systems and the possible dynamics, respectively. In the first part of this paper, we present a result on separation of simplices and balls (up to affine equivalence) among all compact convex sets in two- and three-dimensional Euclidean spaces, which focuses on the set of extreme points and the action of affine transformations on it. Regarding the above-mentioned axiomatization of quantum physics, our result corresponds to the case of simplest (2-level) quantum system. We also discuss a possible extension to higher dimensions. In the second part, towards generalizations of the framework of general probabilistic theories and several existing results including ones in the first part from the case of compact and finite-dimensional physical systems as in most of the literature to more general cases, we study some fundamental properties of convex sets and affine maps that are relevant to the above subject.

1. Introduction

1.1. Backgrounds and our contributions

Convexity is a ubiquitous notion in mathematics, frequently appearing not only in geometry but also in other various research areas, including many applications to outside mathematics (see e.g., [1,2] and references therein). Among them, an interesting study of convexity has emerged in the foundations of quantum mechanics. These activities aim at interpreting quantum physics as an instance of more general physical theories (called e.g., "general probabilistic theories"), the latter being axiomatized from operational viewpoints, using "(physical) states" and "measurements" as the basic notions. Here, probabilistic mixtures of states are formalized as convex combination of states, therefore the notion of convexity is essential in those studies. A motivation of studying such general theories is to establish a unified theoretical framework to describe quantum (and classical) physics together with its possible variants or generalizations. Potential applications of such activities would include cryptography with long-standing security; if one wants to estimate the security of present cryptographic schemes against attacks using physical devices in the next 100 years, where the present quantum physics may be improved by some advanced theory, then such an observation of general physical theories may be of some help. Another motivation is to characterize quantum physics among such general theories, giving a re-axiomatization of quantum physics

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based on physical principles, which is expected to be more physically intuitive than von Neumann's original axiom based (mysteriously) on Hilbert spaces. For more introduction to this subject from physical viewpoints and several preceding works, see e.g., [3–7] and references therein.

A common philosophy in the above-mentioned studies of general physical theories can be understood as follows: We put mathematical assumptions that are essential (or inevitable) from physical viewpoints, while the quantity of other "technical" assumptions should be as small as possible. To formulate the state space (the set of physical states) of such a general physical theory, the state space is conventionally assumed to be a convex set, reflecting the above-mentioned requirement that any probabilistic mixture of two states should also be a state.

In the first part of this paper, we investigate finite-dimensional compact state spaces (convex sets) equipped with symmetry and some additional special properties. In the theory of convex polytopes, the notion of symmetry (precisely, vertex-transitivity of affine isometric transformation groups) has played one of the most significant roles and such symmetric convex polytopes have been intensively studied (e.g., [8]). Here, as usual, we say that a group G *acts transitively* on a set X, if for any $x_1, x_2 \in X$, we have $g \cdot x_1 = x_2$ for some element $g \in G$. In the studies of general physical theories, the symmetry property [9] can also be considered as one of the physical principles which can be interpreted as a possibility of reversible transformation between pure states [10–12] or as a physical equivalence of pure states [13,6]. In this context, the characterization of a state space under the hypothesis of symmetry becomes an important subject. In this paper, we give the following characterizations on 2- and 3-dimensional compact convex sets with the symmetry property:

Theorem 1. *Let $\mathcal{S}$ be a compact convex subset of a Euclidean space with 2-dimensional affine hull;* $\mathrm{Aff}(\mathcal{S}) = \mathbb{R}^2$*. Then the group of bijective affine transformations of $\mathcal{S}$ acts transitively on the set $\mathcal{S}_{\mathrm{ext}}$ of extreme points of $\mathcal{S}$ (see above for the terminology) if and only if $\mathcal{S}$ is affine isomorphic to one of the following two kinds of objects:*

1. *A symmetric (or vertex-transitive) convex polygon.*
2. *The (2-dimensional) unit disk.*

Theorem 2. *Let $\mathcal{S}$ be a compact convex subset of a Euclidean space with 3-dimensional affine hull;* $\mathrm{Aff}(\mathcal{S}) = \mathbb{R}^3$*. Then the group of bijective affine transformations of $\mathcal{S}$ acts transitively on the set $\mathcal{S}_{\mathrm{ext}}$ of extreme points of $\mathcal{S}$ if and only if $\mathcal{S}$ is affine isomorphic to one of the following three kinds of objects:*

1. *A 3-dimensional symmetric (or vertex-transitive) convex polytope.*
2. *A 3-dimensional circular cylinder* $\{{}^t(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \le 1,\ 0 \le z \le 1\}$.
3. *The 3-dimensional unit ball.*

Here we give a remark on a related work: In a preceding paper [9], Davies also studied the finite-dimensional compact convex sets $\mathcal{S}$ whose groups G of bijective affine transformations act transitively on the sets of extreme points, and presented a classification of the convex sets $\mathcal{S}$ such that its group G of symmetry as above is equal to a given compact group, in terms of classification of finite-dimensional subspaces of the left regular representation of the given compact group. However, in that work any relation between the symmetric convex sets with *different* groups of symmetry, and also the concrete shape of a symmetric convex set induced by each subspace of the regular representation, have not been clarified. Therefore, to classify *all* symmetric convex sets based on that result, it is basically required to determine the concrete shape of *every* convex set constructed from some representation space of *every* compact group. On the other hand, our results above aimed at determining the symmetric convex sets without any restriction for the groups of symmetry.

Next, we investigate another kind of physically motivated hypothesis, called the *spectrality* of state spaces. First, we remind the definition of distinguishability of states in a single shot measurement (see, e.g., [13] for its motivation from physical viewpoints):

Definition 1. We say that points $s_1, s_2, \ldots, s_n$ of a convex subset $\mathcal{S}$ of a real vector space are *distinguishable* if there exists a collection $(e_i)_{i=1}^n$ of n affine functionals $e_i : \mathcal{S} \to \mathbb{R}$ such that $e_i \ge 0$, $\sum_{i=1}^n e_i = 1$ and $e_i(s_i) = 1$ for every $1 \le i \le n$.

Note that any non-empty subset of a set of distinguishable points is also distinguishable. Roughly speaking, if a convex set $\mathcal{S}$ has n distinguishable points $s_1, \ldots, s_n$, then "lossless" encoding of any n-bit information into a point of $\mathcal{S}$ is (in principle) possible by using these n points. (A geometric interpretation of this definition will be supplied in Lemma 2 below.) Then we introduce the following definition:

Definition 2 (*See e.g., [14,15]*)**.** Let $\mathcal{S}$ be a convex subset of a real vector space. We say that $\mathcal{S}$ has *spectrality* (or, as in [6], $\mathcal{S}$ is *distinguishably decomposable*) if each $s \in \mathcal{S}$ admits a decomposition $s = \sum_{j=1}^{\ell} \lambda_j s_j$ $(1 \le \ell < \infty)$ into distinguishable extreme points $s_1, \ldots, s_\ell \in \mathcal{S}_{\mathrm{ext}}$ such that $\sum_{j=1}^{\ell} \lambda_j = 1$ and $\lambda_j \ge 0$ for every j. Moreover, if the number ℓ of distinguishable extreme points in each decomposition is bounded above by k, then we say that $\mathcal{S}$ has *k-spectrality*.

In general physical theories, the spectrality of state spaces can also be one of the physical principles which can be interpreted as the possibility of state preparation with a probabilistic mixture of distinguishable pure states [6]. Here we emphasize that the distinguishable points appearing in Definition 2 should be *extreme points*, which rules out, e.g., the square and any regular polygon since a generic point in these convex sets cannot be decomposed into extreme points (even though such a point can be decomposed into boundary points; see Corollary 2). By using this notion, in this paper we give the