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Fiber averaged dynamics associated with the Lorentz force equation

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ABSTRACT

It is shown that the Lorentz force equation is equivalent to the auto-parallel condition ${}^L\nabla_{\dot{x}}\dot{x} = 0$ of a linear connection ${}^L\nabla$ defined on a convenient pull-back vector bundle. By using a geometric averaging method, an associated *averaged Lorentz connection* $\langle {}^L\nabla\rangle$ and the corresponding auto-parallel equation are obtained. After this, it is shown that in the ultra-relativistic limit and for narrow one-particle probability distribution functions, the auto-parallel curves of $\langle {}^L\nabla\rangle$ remain *nearby* close to the auto-parallel curves of ${}^L\nabla$. Applications of this result in beam dynamics and plasma physics are briefly described.

1. Introduction

Given a four dimensional Lorentzian manifold $(\mathbf{M}, \eta)$, the motion of a classical point charged particle under the influence of an external electromagnetic field and without taking into account radiation reaction is described by the *Lorentz force equation*. This work explores two related topics:

1. A geometric description of the Lorentz force equation and
2. An averaged version of the Lorentz force equation and its relation with the original Lorentz force equation.

Once a geometric version of the Lorentz force equation is available, we can apply the *averaging method* [1]. It is proved that in the *ultra-relativistic limit* and under some natural assumptions on the averaging model, the solutions of the original Lorentz force equation can be approximated by the solutions of an *averaged Lorentz force* equation with high accuracy. This fact has interesting consequences for plasma modeling.

The geometrization of the Lorentz force equation is interesting from the point of view of the analysis of its symmetries and mathematical structure. It turns out that there are several connections whose geodesic equations are the Lorentz force equation (3.6) constrained by the speed normalization (3.7) (see for instance [2] for one of such alternative connections). The structure of the Lorentz force equation (3.6) is described in terms of the Levi-Civita connection ${}^\eta\nabla$, the longitudinal and transverse tensors (see the tensor L and T discussed in Section 3). The transverse tensor does not contribute to the geodesic equation but it does to the motion of a *charged gyroscope*, for example.

The *averaged Lorentz dynamics* has some advantages over the *Lorentz dynamics*. The first and most notorious is that the differential equation $\langle {}^L\nabla\rangle_{\dot{\tilde{x}}}\dot{\tilde{x}} = 0$ is easier to work with than the original Lorentz force equation ${}^L\nabla_{\dot{x}}\dot{x} = 0$. This allows us to perform easier analysis and computations with $\langle {}^L\nabla\rangle_{\dot{\tilde{x}}}\dot{\tilde{x}} = 0$ than with ${}^L\nabla_{\dot{x}}\dot{x} = 0$ (following the general philosophy of the

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averaging method in classical mechanics [3]). This is particularly useful when we apply the model to systems composed of a large number of point charged particles, composing a non-neutral plasma [4,5].

From a topological point of view, the averaged equation contains the same information than the original equation. This is because the connections ${}^L\nabla$ and $\langle {}^L\nabla\rangle$ are connected by a continuous homotopy, which is similar to the convex invariance found in Finsler symmetry [1]. This property could have relevance for the study of the topological properties of confined plasmas, since could allow to study topological properties of plasmas by a simplified averaged model.

The structure of this work is the following. In Section 2, we explain how, given a second order differential equation, several connections are associated to the differential equation.

In Section 3, a geometric interpretation for the Lorentz force equation is explained. In particular we show how to extract from the Lorentz force equation (3.6), a linear connection defined on a convenient pull-back bundle. Such connection will be called the *Lorentz connection* ${}^L\nabla$. Then it is shown that Eq. (3.6) corresponds to the auto-parallel condition of the Lorentz connection.

In Section 4, we follow Refs. [1,6] and we define the *average of a family of automorphisms*, a generalization of the *fiber integration operation* of R. Thom and S.S. Chern [7]. Some examples of averaging in different geometric frameworks are explained. In Section 5, given a manifold $\mathbf{M}$ and a subbundle $\hat{\mathbf{N}}$ of $\mathbf{TM}$, we apply the averaging method to an arbitrary linear connection on the pull-back bundle $\pi^*|_{\hat{\mathbf{N}}}\mathbf{TM}$. The average of such connections are affine connections of the manifold $\mathbf{M}$. The averaging method is applied to the Lorentz connection, obtaining the *averaged Lorentz connection* $\langle {}^L\nabla\rangle$.

In Section 6, the solutions of the Lorentz force equation (3.6) and those of the auto-parallel curves of the averaged connection are compared. The measure used to calculate the momentum moments in the averages are borrowed from relativistic kinetic theory [8]. Then it is proved that in the *ultra-relativistic limit*, for narrow 1-particle probability distribution functions for the same initial conditions, the solutions of the auto-parallel equations ${}^L\nabla_{\dot{x}}\dot{x} = 0$ and $\langle {}^L\nabla\rangle_{\dot{\tilde{x}}}\dot{\tilde{x}} = 0$ remain *near* to each other, even for finite time evolution. We discuss the limits of validity of the result and provide two relevant examples of averaging for the Lorentz force equation.

In Section 7, some applications in beam dynamics and plasma physics of the averaged Lorentz equation are briefly described, as well as other potential applications of the *averaged dynamics* and the extension to other equations of evolution. In Section 8, the hypothesis used in the proofs of the main results of this paper are discussed.

2. Connections and second order differential equations

Let $\pi : \mathbf{TM} \to \mathbf{M}$ be the canonical projection of the tangent bundle $\mathbf{TM}$ and $\Gamma\mathbf{TTM}$ the set of vector fields over $\mathbf{TM}$.

Definition 2.1. A second order differential equation (or semi-spray) is a smooth vector field $G \in \Gamma\,\mathbf{TTM}$ such that $\pi_* G|_u = u,\ \forall\, u \in \mathbf{TM}$.

Remark 2.2. If we require that the integral curves of the semi-spray G are invariant under affine re-parameterization, it is necessary to exclude the origin $0 \in \Gamma\mathbf{TM}$ from the domain of definition of G. This is because the invariance under affine re-parameterization implies that $G(x, y)$ must be homogeneous of degree one in y. The requirement that the integral curves are affine re-parameterization invariant implies that G is a special type of second order differential equation known as spray. Therefore, a spray is not necessarily smooth at the origin. Moreover, since we will apply the geometric formalism to the Lorentz force differential equation, the associated geodesic spray is not defined over the whole $\mathbf{TM}\setminus\{0\}$, but only on the sub-bundle of timelike vectors fields $\mathbf{N} \to \mathbf{M}$ or on the future pointed unit hyperboloid sub-bundle $\mathbf{\Sigma}^+_x \to \mathbf{M}$, as we will define later. The particular geometric setting depends on the choice of parameterization for the curves describing the point particle. Therefore, it is convenient to develop the general geometric formalism for the case that G is defined on an arbitrary open sub-bundle $\hat{\mathbf{N}} \hookrightarrow \mathbf{TTM},\ G \in \Gamma T\hat{\mathbf{N}}$. In this case $\pi|_{\hat{\mathbf{N}}} : \hat{\mathbf{N}} \to \mathbf{M}$ is the restriction to $\hat{\mathbf{N}}$ of the canonical connection $\pi : \mathbf{TM} \to \mathbf{M}$.

2.1. The non-linear connection associated to a second order differential equation

Let us consider the differential map $\pi_*|_{\hat{\mathbf{N}}} : \mathbf{T}\hat{\mathbf{N}} \to \mathbf{TM}$. The vertical bundle (or vertical distribution) is the kernel $\mathcal{V} := \ker(\pi_*|_{\hat{\mathbf{N}}})$. At each point $u \in \hat{\mathbf{N}}$ one has $\ker(\pi_*|\hat{\mathbf{N}}(u)) := \mathcal{V}_u$.

Definition 2.3. A non-linear connection on $\hat{\mathbf{N}}$ is a distribution $\mathcal{H}$ on $\mathbf{T}\hat{\mathbf{N}}$ such that

$$\mathbf{T}_u\hat{\mathbf{N}} = \mathcal{V}_u \oplus \mathcal{H}_u,$$

for each $u \in \hat{\mathbf{N}}$.

Given a semi-spray as in 2.1, there are connections partially characterized by the fact that by the projection $\pi|_{\hat{\mathbf{N}}} : \hat{\mathbf{N}} \to \mathbf{M}$, the auto-parallel curves coincide with the projection of the integral curves of G [9,10]. In order to introduce such connections, let us first consider the complete lift and vertical lift of tangent vectors [9]. The vertical lift y^v of $y \in \mathbf{T}_x\mathbf{M}$ at the point $(x, \tilde{y}) \in \mathbf{T}_x\mathbf{M}$ is the tangent vector at $s = 0$ of the curve $s \mapsto (x, y + s\tilde{y})$. The complete lift of a vertical field $X \in \Gamma\mathbf{TM}$ is a vector field $X^c \in \mathbf{TN}$ whose flow in $\hat{\mathbf{N}}$ is $(s, (x, y)) \mapsto (\phi_s(x), \phi_{s*}(y))$, where $(s, x) \mapsto \phi_s(x)$ is the flow of X and ϕ_{s*} is the