Journal of Geometry and Physics 86 (2014) 94–100

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Classical physics of particles and fields

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ARTICLE INFO

Article history:
Received 19 August 2013
Received in revised form 9 November 2013
Accepted 14 July 2014
Available online 19 July 2014

Keywords:
Geodesics
Canonical distance form
Weyl field

ABSTRACT

We formulate geodesics in terms of a parallel transfer of a particle state vector transformed by local Lorentz and Yang–Mills symmetry groups. This formulation is based on horizontal fields and requires a canonical distance form. Arguments are presented in favour of scaling distance in our space–time with a scalar field.

1. Introduction

Classical physics describes motion of particles under an action of classical fields. Classical particles are usually assumed to be structureless material points. Classical fields are produced by charges that attract or repel each other. It is also conventionally assumed that elementary charges (or simply elementary particles) of classical physics are point-like and have vanishing spatial sizes. (This follows from the fact that classical solutions with charge distributed in some area of space are normally not stable. Hence unknown additional forces are needed to stabilize elementary particles if they were to occupy some finite region in space.) The classical picture therefore contains a space filled with delta-like charges and fields described by field potentials everywhere except points of charge singularities.

It is also widely accepted that classical gauge fields represent connections in a fibre bundle associated with a particle representation transforming under Lorentz and a local symmetry group of particle interactions [1]. There exists an asymmetry in dealing with particle representations and connections in classical physics: connections enter the scheme of classical physics (as field potentials) while particle representations (fibre co-ordinates on which these connections act in an appropriate associated form) often do not. For example, electromagnetic 4-potential (that represents connection in a space of complex particle representations) is an element of classical physics, while complex particle representations are not. As a result, fields lose their geometrical meaning in classical physics and appear to be ad-hoc assumptions of classical dynamics. It seems natural to eliminate the asymmetry and restore geometrical meaning of classical fields by adding an internal structure to a classical particle. Here we discuss a possible generalization of classical physics that incorporates local symmetries of particle interactions. We restrict ourselves to classical physics of particles whose behaviour is defined by a local state associated with a particle "singularity".

2. Preliminaries

Let us suppose that the space of a classical particle is an associated fibre bundle $E(M, F, G, P)$, where M is the base manifold, F is the manifold on which the group G acts on the left (F can be described in terms of particle states $|\phi\rangle$). We

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http://dx.doi.org/10.1016/j.geomphys.2014.07.022
0393-0440/

reformulate geodesics of M in terms of the parallel transport of a particle state vector $|\phi\rangle$ instead of the parallel transfer of a tangent vector. To perform this task, we consider a definition of geodesics based on horizontal fields [2]. A standard horizontal vector field $B(\xi)$ is defined through a set of equations:

$$\begin{cases}\omega(B(\xi)) = 0\\ \theta(B(\xi)) = \xi,\end{cases} \tag{1}$$

where ξ belongs to $\mathbb{R}_n$ (n is the dimension of M). Here, ω and θ are the connection and canonical forms of the principal linear fibre bundle, respectively [2]. Ref. [2] shows that an integral curve of the horizontal vector field B projects onto geodesics under natural projection to M. This provides a definition of geodesics as a solution of a system

$$\begin{cases}\nabla_{\dot{x}} u = \dot{u} + \hat{\omega}(\dot{x}) \cdot u = 0\\ \theta(\dot{x}) = u,\end{cases} \tag{2}$$

where u is an $\mathbb{R}_n$-vector generated by the canonical form $\theta(\dot{x})$ and $\nabla_{\dot{x}} u$, $\hat{\omega}(\dot{x})$, $\theta(\dot{x})$ are the interior products $\iota_{\dot{x}}\nabla u$, $\iota_{\dot{x}}\omega$, $\iota_{\dot{x}}\theta$. Locally (on an open subset $U \subset M$) the principal fibre bundle $P(M, G)$ can be trivialized as $P(M, G) \sim M \times G$ and hence expressed as $P = (\pi(p), \varphi(p))$, where $p \in P$, $\pi(p)$ is the projection on M and $\varphi(p)$ is the fibre co-ordinates. This gives natural cross-section $\sigma = (\pi(x), e)$ which allows one to project forms of the principal fibre bundle onto M. In this projection, the linear canonical form of a manifold (aka the fundamental form, aka the solder form) is given in Cartan notations [3] as $\theta = \theta^i e_i$, where e_i are basis vectors of the tangent space and θ^i are the canonical one-forms of the linear connection [2]. For natural coordinates x^i, the canonical form of linear connection is $\theta^i = dx^i$ and $e_i = \frac{\partial}{\partial x^i}$. Then, the second equation of (2) is $u^i = \dot{x}^i$. Hence we can eliminate the tangent vector u and write the geodesics equations in the conventional form

$$\ddot{x}^i + \omega^i_j(\dot{x})\dot{x}^j = 0. \tag{3}$$

It is worth noting that the system (2) has transparent physical meaning. The first equation of (2) implies that a tangent vector u is transported parallel along a geodesic curve. The second equation of the system (2) determines a displacement on the manifold for a specific value of a tangent vector, or, in other words, solders the manifold with the tangent space.

First, we rewrite geodesic equations in terms of representations $|\phi\rangle$ of the Lorentz group $SO^+(1, n-1)$. For concreteness, we consider the case $n = 4$. (Our results can be easily generalized to other dimensions and other orthogonal groups.) To perform the task we have to reduce the linear connection of arbitrary frames to the linear connection of the orthonormal frame rotations since only orthonormal frames support the action of the Lorentz group. Let e_a be a section of orthogonal frames so that $e_i = e_a e^a_i$, where e^a_i are n-ads (n is the space dimension), and ω^a_b be the connection forms ω^i_j reduced to the orthonormal frames e_a: $\omega^a_b = e^a_j \omega^j_i e^i_b + e^a_i de^i_b$, where e^i_b is the inverse to n-ads. We describe the particle state using a representation vector $|\phi\rangle$ assigned to the particle position so that $|\phi(\tau)\rangle$, where τ is the curve parameter. (In the conventional definition of geodesics $|\phi\rangle$ is the tangent vector u assigned to the particle position on the manifold.) Then, the first equation of the system (2) has a straightforward generalization to a representation of the Lorentz group (in local co-ordinates)

$$\nabla_{\dot{x}} |\phi\rangle = d|\phi\rangle / d\tau + \hat{\omega}(\dot{x}) |\phi\rangle = 0. \tag{4}$$

Here $|\phi\rangle$ is a vector of the representation, $\hat{\omega} = (1/2)\hat{G}_{[ab]}\omega^{ab}$ is the linear connection form in the representation, $\hat{G}_{[ab]}$ are the generators of the Lorentz group in the same representation [1]. Eq. (4) simply tells that any vector object associated with a geodesic curve is transported parallel along it and we will consider it to be valid for representations of a local Yang–Mills group as well.

The difficulty lies with the second equation of the geodesic system (2) (that calculates the direction and magnitude of the displacement on a manifold for a specific value of the tangent vector and solders manifold with its tangent space). This equation cannot be formulated in terms of representation vectors since one-to-one correspondence between representation vectors $|\phi\rangle$ and displacements on a manifold is absent in a general case. It has to be said that the second equation of the geodesic system (2) is intuitively obvious to such degree that it is usually taken for granted. However, it is not necessarily evident as it relates two entities of *different geometrical nature*: non-local motion along the curve $x(\tau)$ and a local tangent vector u. For this reason the second equation of (2) has to be amended in order to describe geodesics in terms of representations of local group of symmetries.

3. Geodesics as integral curve of horizontal field in associated fibre bundle

To find a correct relation between displacements on a manifold and representation vectors we note that representation operators are the only geometric objects at our disposal. A fundamental property of a (physical) operator is a set of its eigenvectors lying in the representation space. Thus, an operator valued one-form (which maps a displacement to a space of operators) would establish a mapping of a manifold displacement to eigenvectors of a corresponding operator. This mapping is multivalued since different eigenvectors may correspond to the same displacement. However, being taken with Eq. (4), this correspondence will lead to a unique curve in the fibre bundle associated with the representation and yield a geodesic as a projection of this curve.