Journal of Geometry and Physics 86 (2014) 112–121

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

On weak Berwald (α, β)-metrics of scalar flag curvature

Guangzu Chen *, Qun He, Shengliang Pan

Department of Mathematics, Tongji University, Shanghai 200092, PR China

ARTICLE INFO

Article history:
Received 26 October 2013
Accepted 22 July 2014
Available online 30 July 2014

MSC:
53B40
53C60

Keywords:
(α, β)-metric
Weak Berwald Finsler metric
Flag curvature
Berwald metric
Minkowskian metric

ABSTRACT

In this paper, we study weak Berwald (α, β)-metrics of scalar flag curvature. We prove that this kind of (α, β)-metrics must be Berwald metric and their flag curvatures vanish. In this case, they are locally Minkowskian.

© 2014 Elsevier B.V. All rights reserved.

1. Introduction

It is well known that the spray coefficients G^i of a Riemannian metric are quadratic in $y \in T_xM$. It is a natural question whether or not there are non-Riemannian metrics whose spray coefficients G^i are quadratic in y. There are plenty of such Finsler metrics firstly investigated by L. Berwald. Thus we call Finsler metrics whose spray coefficients are quadratic in y Berwald metrics.

Let F be a Finsler metric on an n-dimensional manifold M and G^i be the geodesic coefficients of F, consider the following quantity

$$B_j{}^i{}_{kl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

We obtain a well-defined tensor $\mathbf{B} := B_j{}^i{}_{kl} dx^j \otimes dx^k \otimes dx^l \otimes \partial_i$ on $TM/\{0\}$. $\mathbf{B}$ is called Berwald curvature. It is clear that F is a Berwald metric if and only if $\mathbf{B} = 0$. Define the mean Berwald curvature $\mathbf{E} := E_{ij} dx^i \otimes dx^j$ by

$$E_{ij} := \frac{1}{2} B_m{}^m{}_{kl}.$$

F is called a weak Berwald metric if $\mathbf{E} = 0$. It is obvious that Berwald metrics must be weak Berwald metrics but the reverse is not true.

The flag curvature, a natural extension of the sectional curvature in Riemannian geometry, plays the central role in Finsler geometry. Generally, the flag curvature depends not only on the section but also on the flagpole. A Finsler metric is of scalar

* Corresponding author. Tel.: +86 13524345535.
E-mail addresses: chenguangzu1@163.com (G. Chen), hequn@tongji.edu.cn (Q. He), slpan@tongji.edu.cn (S. Pan).

http://dx.doi.org/10.1016/j.geomphys.2014.07.031
0393-0440/© 2014 Elsevier B.V. All rights reserved.

flag curvature if its flag curvature depends only on the flagpole. It is one of hot and difficult problems to characterize Finsler metrics with scalar flag curvature.

In Finsler geometry, there is an important class of Finsler metrics expressed in the following form

$$F = \alpha\phi(s), \quad s = \frac{\beta}{\alpha},$$

where $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemann metric and $\beta = b_i(x)y^i$ is a 1-form with $b := \|\beta_x\|_\alpha = \sqrt{a^{ij}b_ib_j} < b_0$. It is proved that $F = \alpha\phi(\beta/\alpha)$ is a positive definite Finsler metric if and only if the function $\phi = \phi(s)$ is a C^∞ positive function on an open interval $(-b_0, b_0)$ satisfying [1]

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \le b < b_0.$$

Such a metric is called an (α, β)-metric. In particular, when $\phi = 1 + s$, Finsler metric $F = \alpha + \beta$ is Randers metric with $\|\beta\|_\alpha < 1$. More generally, when $\phi = k_1\sqrt{1 + k_2s^2} + k_3s$, where $k_1 > 0$, k_2 and $k_3 \ne 0$ are constant, Finsler metrics $F = k_1\sqrt{\alpha^2 + k_2\beta^2} + k_3\beta$ are called Finsler metrics of Randers type.

Let

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \qquad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}),$$

$$b^i := a^{ij}b_j, \qquad s_i := b^js_{ji}, \qquad s^i{}_j := a^{il}s_{lj}, \qquad r_i := b^lr_{li},$$

$$s_0 := s_iy^i, \qquad s^i{}_0 := s^i{}_jy^j, \qquad r_{00} := r_{ij}y^iy^j$$

where "|" denotes the horizontal covariant derivative with respect to α.

In [2], Z. Shen shows that any regular (α, β)-metrics are Berwald metric if and only if β is parallel with respect to α, that is, $b_{i|j} = 0$. However, how to characterize the weak Berwald (α, β)-metrics is still open. In 2009, X. Cheng and C. Xiang study a class of (α, β)-metrics in the form $F = (\alpha + \beta)^{m+1}/\alpha^m$, where $m \ne -1, 0, -1/n$ and prove that F is weak Berwald if and only if it satisfies that β is a killing 1-form with constant length with respect to α, that is, $r_{ij} = s_i = 0$ [3]. In [4], X. Cheng and C. Lu show that two kinds of weak Berwald (α, β)-metrics in the form $F = \alpha + \epsilon\beta + k(\beta^2/\alpha)$ (ϵ and $k \ne 0$ are constants) and $F = \alpha^2/(\alpha - \beta)$ are weak Berwald if and only if they satisfy that β is a killing 1-form with constant length with respect to α. Further, they prove that the two kinds of weak Berwald (α, β)-metrics with scalar flag curvature must be locally Minkowskian. In this paper, we have

Theorem 1.1. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$ be a non-Riemannian (α, β)-metric on a manifold M of dimension $n \ge 3$. Suppose $\phi \ne k_1\sqrt{1 + k_2s^2} + k_3s$ for any constant $k_1 > 0$, k_2 and $k_3 \ne 0$. Then F is a weak Berwald metric if and only if β is a killing 1-form with constant length with respect to α.*

Further, we obtain

Theorem 1.2. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$ be a non-Riemannian (α, β)-metric of scalar flag curvature on a manifold M of dimension $n \ge 3$. Suppose $\phi \ne k_1\sqrt{1 + k_2s^2} + k_2s$ for any constant $k_1 > 0$, k_2 and $k_3 \ne 0$ on M. Then F is a weak Berwald metric if and only if F is a Berwald metric and the flag curvature $\mathbf{K} = 0$. In this case, F must be locally Minkowskian.*

2. Preliminaries

For a given Finsler $F = F(x, y)$, the geodesics of F are characterized locally by a system of 2nd ODEs as follows [1],

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

where

$$G^i = \frac{1}{4}g^{il}\left\{[F^2]_{x^my^l}y^m - [F^2]_{x^l}\right\}.$$

G^i are called the *geodesic coefficients* of F.

There are many interesting non-Riemannian quantities in Finsler geometry. For a non-zero vector $y \in T_pM$, the Cartan torsion $\mathbf{C}_y = C_{ijk}dx^i \otimes dx^j \otimes dx^k : T_pM \otimes TpM \otimes TpM \longrightarrow R$ is defined by

$$C_{ijk} := \frac{1}{4}[F^2]_{y^iy^jy^k} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}(x, y).$$

The mean Cartan torsion $\mathbf{I}_y = I_i(x, y)dx^i : T_pM \longrightarrow R$ is defined by

$$I_i := g^{jk}C_{ijk},$$

where $(g^{ij}) := (g_{ij})^{-1}$ and $g_{ij} := \frac{1}{2}[F^2]_{y^iy^j}$. It is obvious that $C_{ijk} = 0$ if and only if F is Riemannian. According to Deicke's theorem [5], a Finsler metric is Riemannian if and only if the mean Cartan torsion vanishes.

Download English Version:

https://daneshyari.com/en/article/1894765

Download Persian Version:

https://daneshyari.com/article/1894765

Daneshyari.com