Journal of Geometry and Physics 86 (2014) 273–283

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Generalized holomorphic structures

Yicao Wang

Department of Mathematics, Hohai University, Nanjing 210098, China

ARTICLE INFO

Article history:
Received 20 June 2013
Received in revised form 10 June 2014
Accepted 18 August 2014
Available online 26 August 2014

MSC:
53D18
53D05
53C15

Keywords:
Principal bundle
Generalized complex structure
Generalized holomorphic structure
Group action
Reduction

ABSTRACT

We define the notion of generalized holomorphic principal bundles and establish that their associated vector bundles of holomorphic representations are generalized holomorphic vector bundles defined by M. Gualtieri. Motivated by our definition, several examples of generalized holomorphic structures are constructed. A reduction theorem of generalized holomorphic structures is also included.

1. Introduction

In generalized complex (GC for short) geometry initiated by N. Hitchin [1] and further developed by M. Gualtieri [2,3] and others, generalized holomorphic (GH for short) structures are the analogue of holomorphic structures in classical complex geometry. They are special examples of Lie algebroid modules, and include some already known geometric objects, e.g. holomorphic Poisson modules.

In general, unlike its complex-geometric counterpart, it is not easy to construct nontrivial GH structures. The existing examples in the literature are flat bundles over symplectic manifolds, co-Higgs bundles over complex manifolds [4–7], Poisson modules over holomorphic Poisson manifolds [8,9]. In [9], Hitchin also adapted a differential geometric version of the Serre construction in algebraic geometry to produce rank-2 GH vector bundles over a compact connected GC 4-manifold whose type change locus has a nondegenerate component.

In [10], from a viewpoint of deformations of GC structures, the author investigated some aspects of GH vector bundles. This paper is then a continuation of that one, but from a different viewpoint: we extend the notion of holomorphic principal bundles to the generalized setting Definition 4.4. In complex geometry, there are three equivalent ways to define a holomorphic principal bundle: by a 1-cocycle of holomorphic transition functions valued in a complex Lie group, by an equivariant complex structure in the total space and a holomorphic projection, or by a complex distribution in the total space. However, the third way is the one we choose to generalize—it seems that no suitable notions of GH functions and GH maps in the literature can be used to generalize the notion of holomorphic principal bundles in the other two ways. So we define the notion of GH principal bundles in terms of (generalized) distributions. Another ingredient to define this notion is the reduction theory of Courant algebroids and Dirac structures [11]. It is adapted to apply to a generalized distribution in the total space which is not maximal—the reduced generalized distribution is precisely the GC structure in the base manifold.

E-mail addresses: yicwang@mail.ustc.edu.cn, yicwang@hhu.edu.cn.

http://dx.doi.org/10.1016/j.geomphys.2014.08.014
0393-0440/

After defining the notion of GH principal bundles, we establish the relation between GH principal bundles and GH vector bundles—the associated vector bundle of a GH principal bundle and a holomorphic representation of the structure group is a GH vector bundle canonically Theorem 4.13. This is the analogue of the well-known relation between holomorphic principal bundles and holomorphic vector bundles. Therefore, our notion of GH principal bundles provides another possible way to construct GH structures.

The paper is organized as follows. In Section 2, we recall the basic and necessary knowledge of GC geometry and the reduction theory of Courant algebroids and Dirac structures. In Section 3, we briefly investigate some properties of GH vector bundles and show that co-Higgs bundles are the basic local ingredient of such bundles, at least around a regular point Propositions 3.1 and 3.2. Motivated by this observation, and starting with a co-Higgs bundle, we construct an example of GH vector bundle, which is not a co-Higgs bundle. In Section 4, we define the notion of GH principal bundles and explore some examples in this context, e.g. co-Higgs principal bundles and Poisson principal bundles (they are the counterparts of co-Higgs vector bundles and Poisson modules). The relation between GH principal bundles and GH vector bundles is also studied there Proposition 4.12 and Theorem 4.13. We also show the particularity of GH principal $\mathbb{C}^*$-bundles, i.e. the total space of a GH principal $\mathbb{C}^*$-bundle acquires a canonical GC structure Theorem 4.14. In Section 5, under certain compatibility conditions, we prove that a GH principal bundle over a manifold with symmetries descends to another GH principal bundle over the quotient Theorem 5.3. An illustrative example is also given.

2. Background of generalized geometry

We recall some preliminary material of GC geometry, of which the basic references are [2,3,8,11]. In the paper, M is a smooth connected $2m$-manifold and all Lie groups involved are connected.

Generalized geometry is the geometry related to the generalized tangent bundle $\mathbb{T}M := TM \oplus T^*M$, or more generally, an exact Courant algebroid E over M. We follow [11] for the axioms defining a Courant bracket $[\cdot,\cdot]_c$ and all Courant algebroids in the paper refer to exact ones.

Given E, one can always find an isotropic right splitting $s : TM \to E$, which has a curvature form $H \in \Omega^3_{cl}(M)$ defined by

$$H(X, Y, Z) = 2\langle [s(X), s(Y)]_c, s(Z)\rangle, X, Y, Z \in \Gamma(TM).$$

Via the bundle isomorphism $s + \frac{1}{2}\pi^* : TM \oplus T^*M \to E$, the Courant algebroid structure can be transported onto $\mathbb{T}M$. Then the pairing $\langle\cdot,\cdot\rangle$ on E is the natural one, i.e. $\langle X + \xi, Y + \eta\rangle = \frac{1}{2}(\xi(Y) + \eta(X))$, and the bracket is

$$[X + \xi, Y + \eta]_H = [X, Y] + L_X\eta - \iota_Y d\xi + \iota_Y\iota_X H,$$

called the H-twisted Courant bracket. Different splittings are related by B-field transforms, i.e. $e^B(X + \xi) = X + \xi + \iota_X B$, where B is a 2-form.

An isotropic subbundle $\mathfrak{A} \subset E$ is called a generalized distribution and called integrable if it is involutive w.r.t. the Courant bracket. An integrable maximal generalized distribution $\mathfrak{D}$ is called a Dirac structure. These notions can all be extended to the complexified case and what interest us here are those complex Dirac structures called GC structures, i.e. complex Dirac structures L such that $L \oplus \bar{L} = E_{\mathbb{C}}$.[1]

Two extremal GC structures are symplectic and complex structures. Let $E = \mathbb{T}M$ with $H = 0$. If ω is a symplectic structure, then $L = \{X - i\omega(X) | X \in T_{\mathbb{C}}M\}$; if J is a complex structure, then $L = T_{0,1} \oplus T^*_{1,0}$. A more complicated example is a holomorphic Poisson manifold. Let β be a holomorphic Poisson structure on a complex manifold (M, J). Then $L = \{X + \xi + \beta(\xi) | X + \xi \in T_{0,1} \oplus T^*_{1,0}\}$.

Local GC geometry can already be nontrivial: the dimension of $\ker\pi|_L$, called the type, may vary along some subset of M; If it does not change around a point x, x is called regular. Around such an x, up to diffeomorphism and B-field transform, L is precisely the product of a symplectic structure and a complex structure (the transverse complex structure) [3].

If a right splitting is chosen, then $E \cong (\mathbb{T}M, H)$, and the bundle $\mathcal{S}$ of forms can be viewed as the spin bundle of $\mathbb{T}M$; in particular, a GC structure L is characterized by a line bundle $l \subset \mathcal{S}_{\mathbb{C}}$ (the canonical line bundle): L is precisely the annihilator of l under the Clifford action $(X+\xi)\cdot\eta = \iota_X\eta + \xi\wedge\eta$, and integrability of L means, for any $\eta \in \Gamma(l)$, there exists $A \in \Gamma(\mathbb{T}_{\mathbb{C}}M)$ such that

$$d_H\eta := d\eta - H \wedge \eta = A \cdot \eta.$$

Via the pairing and the bracket, a differential operator $d_L : \Gamma(\wedge^k \bar{L}) \to \Gamma(\wedge^{k+1} \bar{L})$ can be defined: for $\sigma \in \Gamma(\wedge^k \bar{L}), X_i \in \Gamma(L)$,

$$d_L\sigma(X_0, \ldots, X_k) = \sum_i (-1)^i \pi(X_i)\sigma(X_0, \ldots, \hat{X}_i, \ldots, X_k) + \sum_{i<j} (-1)^{i+j}\sigma([X_i, X_j]_c, X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k). \quad (2.1)$$

Since L is involutive, $d_L^2 = 0$. The analogue of holomorphic structures in complex geometry is defined as follows.

[1] We denote the complexification of a real space (or bundle) R by $R_{\mathbb{C}}$.