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Geometry of linear and angular momenta of N-particles in the Riemannian space forms and de Sitter space

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ABSTRACT

The total linear and angular momenta are the conserved quantities for the motions of N-body problem. We are concerned with the geometry of the tangential (or normal) lines for the orbit curves of the motions of N-particles. We investigate when such N-tangential (or normal) lines meet at a point in the ambient space, where we consider 2-dimensional Riemannian space form or de Sitter space as the ambient space. We have three applications. The first one is to give the unified interpretation for the existence of the various centers of the triangles, and the second is to obtain the spherical Desargues' theorem. The third is to answer the question when N-geodesics in Riemann surface meet at a point.

1. Introduction

The N-body problem is

$$m_j\ddot{q}_j = -\frac{\partial U}{\partial q_j}, \quad (j = 1, 2, \dots, N)$$

where $q_j = q_j(t) = (x_j(t), y_j(t), z_j(t)) \in \mathbb{R}^3$ and $U = -\sum_{i<j} \frac{m_i m_j}{|q_i - q_j|}$ is the potential energy. Denote by $K(\dot{q})$ the total kinetic energy $\frac{1}{2}\sum_{j=1}^{n} m_j|\dot{q}_j|^2$. Consider the Lagrangian $L(q, \dot{q}) = K(\dot{q}) - U(q)$. Then, the Euler–Lagrange equation for L

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}(q(t), \dot{q}(t))\right) = \frac{\partial L}{\partial q_j}(q(t), \dot{q}(t)), \quad j = 1, 2, \dots, N$$

is equivalent to the N-body problem. Thus, such $q = (q_1, q_2, \dots, q_N)$ is a critical point of the action $\mathcal{A}(q) = \int_{t_0}^{t_1} L(q, \dot{q})dt$. Therefore, the variational method seems to be useful for solving N-body problem.

In [1], Chenciner and Montgomery considered the three body problem for the 3-particles with equal masses in the plane. In particular, they verified the existence of a figure-eight solution to the planer equal-masses three-body problem. They

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call this solution the *figure-eight solution*. However, the explicit solution to this problem has not been discovered yet. It is also unknown even whether the solutions are algebraic or not. On the other hand, by paying attention to the fact that the figure-eight solution has zero angular momentum, Fujiwara, Fukuda and Ozaki [2] found that three tangent lines at the three bodies meet a point at each instant. In fact, this fact holds in more general situation. Consider the motions of three particles with equal-masses 1 in the xy-plane. Denote by l_k (resp. by n_k) the tangential line (resp. normal line) of the curve of kth particle at the point $(x_k(t), y_k(t))$ for any time t, where $k = 1, 2, 3$. Under these situations, Fujiwara et al. implemented a systematic study and proved the following theorem.

Theorem A *([2,3])*.

(1) *If both the linear momentum and the angular momentum of 3-particles vanish, then either the three tangential lines l_1, l_2, l_3 meet at a point or the three tangential lines are parallel to each other.*
(2) *If the moment of inertia is constant and the linear momentum of 3-particles vanish, then either the three normal lines n_1, n_2, n_3 meet at a point or the three normal lines are parallel to each other.*

In particular, the case where all the orbit curves for the motions of the three particles completely coincide and the three particles are on a closed plane curve is interesting. Such an example surely exists and the corresponding closed curve can be described by the Jacobi elliptic functions with some modulus. In fact, in [2] it is proved that the figure-eight solution which satisfies the assumption (1) of Theorem A exists on the lemniscate.

In this paper, we develop the geometry of linear and angular momenta of the motions of N-particles in various ambient spaces, which generalizes the results of Fujiwara et al. stated above.

Let M be a Lagrangian submanifold of $(\mathbb{R}^{2N}, \omega = \sum_{i=1}^{N} dx_i \wedge dy_i)$. Let J be the almost complex structure on $\mathbb{R}^{2N}$, that is, J satisfies the relations $J(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}, J(\frac{\partial}{\partial y_i}) = -\frac{\partial}{\partial x_i}$. Let θ and ϕ be 1-forms on M defined by

$$\theta = \sum_{i=1}^{N}(x_i dy_i - y_i dx_i), \qquad \phi = \sum_{i=1}^{N}(x_i dx_i + y_i dy_i),$$

respectively. Let L be a J-invariant 2-plane through the origin. For each point p on M, we define the dimensions $d_L(p)$ and $e_L(p)$ as follows:

$$d_L(p) = \dim\{v \in T_pM \mid \theta_p(v) = 0, v \perp L\},$$
$$e_L(p) = \dim\{v \in T_pM \mid \phi_p(v) = 0, v \perp L\}.$$

We regard T_pM as the affine space through p and denote it by $H(p)$. We then have the following theorem.

Theorem 1.1. *$H(p)$ and L (resp. $J(H(p))$ and L) are contained in some $(N+2)$-dimensional linear subspace if and only if $d_L(p) \geqq N-2$ (resp. $e_L(p) \geqq N-2$).*

Choose L in Theorem 1.1 as L_0 defined by

$$L_0 = \left\{(x_1, y_1, x_2, y_2, \ldots, x_N, y_N) \in \mathbb{R}^{2N} \mid x_1 = x_2 = \cdots = x_N, y_1 = y_2 = \cdots = y_N\right\}.$$

We have an N-tangential or N-normal lines theorem as a corollary of Theorem 1.1.

Corollary 1.2. *Consider the motions of N-particles in the xy-plane:*

$$c_1(t) = (x_1(t), y_1(t)),\ c_2(t) = (x_2(t), y_2(t)), \ldots, c_N(t) = (x_N(t), y_N(t)).$$

Let M be a Lagrangian submanifold of $\mathbb{R}^{2N}$ defined by $M = c_1 \times c_2 \times \cdots \times c_N$. Let $p = (c_1(t_1), c_2(t_2), \ldots, c_N(t_N))$ be any point of M. Then, $d_L(p) \geqq N-2$ (resp. $e_L(p) \geqq N-2$) if and only if either the N-tangential lines (resp. N-normal lines) in the xy-plane at p meet at a point in the xy-plane or all the N-tangential lines (resp. N-normal lines) in the xy-plane are parallel to each other.

This Corollary implies Theorem A of Fujiwara et al. because the linear momentum(denoted by $\overrightarrow{m}$), the angular momentum(denoted by ω) and the moment of inertia(denoted by I) are respectively given by

$$\overrightarrow{m} = \left(\sum_{i=1}^{3}\frac{dx_i}{dt}, \sum_{i=1}^{3}\frac{dy_i}{dt}\right), \qquad \omega = \sum_{i=1}^{3}\left(x_i\frac{dy_i}{dt} - y_i\frac{dx_i}{dt}\right), \qquad I = \sum_{i=1}^{3}(x_i^2 + y_i^2).$$

We choose a v in Theorem 1.1 by the following $v = \sum_{i=1}^{3}\frac{dx_i}{dt}\frac{\partial}{\partial x_i} + \sum_{i=1}^{3}\frac{dy_i}{dt}\frac{\partial}{\partial y_i}$. We then easily see that $v \perp L$ if and only if $\overrightarrow{m} = \overrightarrow{0}$, $\theta(v) = 0$ if and only if $\omega = 0$, and $\phi(v) = 0$ if and only if $dI(v) = 0$. Therefore, Corollary 1.2 implies the results of Fujiwara et al. The proof of Theorem 1.1 is given in the Section 2. In Sections 3 and 4, we give a generalization of Theorem 1.1 to the case where the ambient space is 2-dimensional non-flat Riemannian space form or de Sitter space.

Let $\mathbb{R}^{3N}$ be a 3n-dimensional Euclidean space. We endow it the metric

$$g_c = \sum_{j=1}^{N}\left((dx_j)^2 + (dy_j)^2 + c(dz_j)^2\right),$$