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A note on diastatic entropy and balanced metrics

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ABSTRACT

We give an upper bound $\mathrm{Ent_d}(\Omega, g) < \lambda$ of the diastatic entropy $\mathrm{Ent_d}(\Omega, g)$ (defined by the author in Mossa (2012) of a complex bounded domain (Ω, g) in terms of the balanced condition (in Donaldson terminology) of the Kähler metric λg. When (Ω, g) is a homogeneous bounded domain we show that the converse holds true, namely if $\mathrm{Ent_d}(\Omega, g) < 1$ then g is balanced. Moreover, we explicitly compute $\mathrm{Ent_d}(\Omega, g)$ in terms of Piatetski-Shapiro constants.

1. Introduction and statements of the main results

Let (M, g) be a real analytic Kähler manifold with associated Kähler form ω. Fix a local coordinate system $z = (z_1, \ldots, z_n)$ on a neighborhood U of a point $p \in M$ and let $\phi : U \to \mathbb{R}$ be a real analytic Kähler potential (i.e $\omega_{|U} = \frac{i}{2}\partial\bar{\partial}\phi$). By shrinking U, we can assume that the potential $\phi(z)$ can be analytically continued to $U \times U$. Denote this extension by $\phi(z, \overline{w})$. Calabi's *diastasis function* $\mathcal{D} : U \times U \to \mathbb{R}$ (E. Calabi [1]) is defined by

$$\mathcal{D}(z, w) = \mathcal{D}_w(z) = \phi(z, \bar{z}) + \phi(w, \overline{w}) - \phi(z, \overline{w}) - \phi(w, \bar{z}).$$

Assume that ω has global diastasis function $\mathcal{D}_p : M \to \mathbb{R}$ centered at p. The *diastatic entropy* at p is defined as

$$\mathrm{Ent_d}(M, g)(p) = \min\left\{c > 0 \mid \int_M e^{-c\,\mathcal{D}_p}\frac{\omega^n}{n!} < \infty\right\}. \tag{1}$$

This definition does not depend on the point p fixed, provided that for every $p \in \Omega$ the diastasis $\mathcal{D}_p$ is globally defined (see [2, Proposition 2.2]). The concept of diastatic entropy was defined by the author in [2] (following the ideas developed in [3]), where he obtained upper and lower bound for the first eigenvalue of a real analytic Kähler manifold.

In this paper we study the link between the diastatic entropy $\mathrm{Ent_d}(\Omega, g)$ and the balanced condition of the metric g. S. Donaldson in [4], in order to obtain a link between the constant scalar curvature condition on a Kähler metric g and the Chow stability of a polarization L, gave the definition of a balanced metric on a compact manifold. Later, this definition has been generalized, by C. Arezzo and A. Loi in [5] (see also [6]), to the noncompact case.

In this paper we are interested in complex domains $\Omega \subset \mathbb{C}^n$ (connected open subset of $\mathbb{C}^n$) endowed with a real analytic Kähler metric g. Let ω be the Kähler form associated to g. Assume that ω admits a global Kähler potential $\phi : \Omega \to \mathbb{R}$. We can define the weighted Hilbert space $\mathcal{H}_\phi$ of square integrable holomorphic functions on Ω with weight $e^{-\phi}$

$$\mathcal{H}_\phi = \left\{s \in \mathrm{Hol}(\Omega) : \int_\Omega e^{-\phi}|s|^2\frac{\omega^n}{n!} < \infty\right\}. \tag{2}$$

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If $\mathcal{H}_\phi \neq \{0\}$ we can pick an orthonormal basis $\{s_j\}_{j=1,\ldots,N\leq\infty}$,

$$\int_\Omega e^{-\phi}\, s_j\, \bar{s}_k\, \frac{\omega^n}{n!} = \delta_{jk}$$

and define its reproducing kernel by

$$K_\phi\,(z, \overline{w}) = \sum_{j=0}^{N} s_j\,(z)\; \overline{s_j\,(w)}. \tag{3}$$

The so called ε*-function* is defined by

$$\varepsilon_g\,(z) = e^{-\phi(z)}\, K_\phi\,(z,\, \bar{z})\,, \tag{4}$$

as suggested by the notation ε_g depends only on the metric g and not on the choice of the Kähler potential ϕ (see, for example, [7, Lemma 1] for a proof).

Definition 1. The metric g is *balanced* if and only if the function ε_g is a positive constant.

In the literature the constancy of ε_g was studied first by J.H. Rawnsley in [8] under the name of η*-function*, later renamed as ε*-function* in [9]. It also appears under the name of *distortion function* for the study of Abelian varieties by J.R. Kempf [10] and S. Ji [11], and for complex projective varieties by S. Zhang [12]. (See also [13] and references therein.)

The following theorem represents the first result of this paper.

Theorem 1. *Let $\Omega \subset \mathbb{C}^n$ be a complex domain and g a Kähler metric such that $\lambda\, g$ is balanced for some $\lambda > 0$. Then the diastatic entropy* $\mathrm{Ent_d}\,(\Omega,\, g)$ *is constant and*

$$\mathrm{Ent_d}\,(\Omega,\, g) < \lambda.$$

In particular (see Lemma 4) the diastatic entropy of a homogeneous bounded domain is constant.

It is natural to ask when the converse of the previous theorem holds true. In the next theorem we show that this is the case when $(\Omega,\, g)$ is assumed to be homogeneous.

Theorem 2. *Let $(\Omega,\, g)$ be a homogeneous bounded domain, then g is balanced if and only if*

$$\mathrm{Ent_d}\,(\Omega,\, g) < 1.$$

We also compute the diastatic entropy of a homogeneous bounded domain in terms of the constants p_k, b_k, q_k and γ_k defined by the Piatetski-Shapiro root structure (see (10) and (11) in the Appendix):

Corollary 2. *Let $(\Omega,\, g)$ be a homogeneous bounded domain. Then the diastatic entropy of $(\Omega,\, g)$ is the positive constant given by*

$$\mathrm{Ent_d}\,(\Omega,\, g)\,(z) = \max_{1\leq k\leq r} \frac{1 + p_k + b_k + q_k/2}{\gamma_k}, \quad \forall z \in \Omega.$$

Example 3. If g is the Bergman metric on Ω, then $\gamma_k = 2 + p_k + q_k + b_k$, $k = 1, \ldots, r$ (see [14, Theorem 5.1] or [15, (2.19)]), so by Corollary 2,

$$\mathrm{Ent_d}\,(\Omega,\, g)\,(z) = \max_{1\leq k\leq r} \frac{1 + p_k + b_k + q_k/2}{2 + p_k + q_k + b_k}, \quad \forall z \in \Omega. \tag{5}$$

In particular, when $(\Omega,\, g)$ is a bounded symmetric domain (i.e. is a homogeneous bounded domain and a symmetric space as Riemannian manifold), there exist integers a and b such that

$$p_k = (k-1)\,a, \qquad q_k = (r-k)\,a, \qquad b_k = b, \qquad \gamma_k = (r-1)\,a + b + 2,$$

where r is the rank of $(\Omega,\, g)$. Therefore, denoted by γ the genus of Ω, by (5) we obtain

$$\begin{aligned}\mathrm{Ent_d}\,(\Omega,\, g)\,(z) &= \max_{1\leq k\leq r} \frac{1 + (k-1)\,a + b + (r-k)\,a/2}{\gamma} \\ &= \frac{1 + (r-1)\,a + b}{\gamma} = \frac{\gamma - 1}{\gamma}, \quad \forall z \in \Omega.\end{aligned}$$

See also [16] for a similar formula relating the volume entropy with the invariants a, b, and r. The table below summarizes the numerical invariants and the dimension of irreducible bounded symmetric domains according to their type (for a more detailed description of these invariants, see e.g. [17,12]). See Fig. 1.