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A family of non-restricted $D = 11$ geometric supersymmetries

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ABSTRACT

We construct a two parameter family of eleven-dimensional indecomposable Cahen–Wallach spaces with irreducible, non-flat, non-restricted geometric supersymmetry of fraction $\nu = 3/4$. Its compactified moduli space can be parameterized by a compact interval with two points corresponding to two non-isometric, decomposable spaces. These singular spaces are associated to a restricted $N = 4$ geometric supersymmetry with $\nu = 1/2$ in dimension six and a non-restricted $N = 2$ geometric supersymmetry with $\nu = 3/4$ in dimension nine.

1. Introduction

In this text we describe in detail the geometric supersymmetry of a family of eleven dimensional manifolds. Geometric supersymmetry is by definition an extension of the Lie algebra of Killing vector fields to a super Lie algebra by geometric data. This roughly means that the odd part of the superalgebra is given by a linear subspace of the sections in a bundle over the manifold compatible with the Killing vector fields—we will be more detailed in the beginning of Section 5. Although the manifolds we consider in this text are homogeneous spaces and, therefore, it would be sufficient to discuss the structure at one point, we will give all results in terms of local coordinates. On the one hand, we do this to emphasize the geometric nature and, on the other hand, because the local description may give an idea for similar constructions in the non homogeneous situation. However, the calculations are similar in both concepts.

First we will provide the general setup of the manifolds that we consider, the Cahen–Wallach spaces. We describe the local structure of the metric and determine the Killing vector fields that yield the even part of our structure, see Sections 2 and 3. Then we turn to the odd part: It will be spanned by sections in a spinor bundle that are parallel with respect to a given connection. Again, we will give the local description and show how this depends on the elements at one point, see Section 4. These preparations lead to Section 5 where we first give a short introduction to geometric superalgebras and geometric supersymmetries. Then we describe in detail how the ingredients provided so far define a geometric superalgebra. In particular, we prove several compatibility conditions. In Section 6 we discuss whether there are situations in which the

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geometric superalgebra yields geometric supersymmetry. We formulate the obstruction and provide a full list. In Section 7 we discuss the moduli space of geometric superalgebras and geometric supersymmetries. Furthermore, we associate the singularities to extended geometric supersymmetries in dimensions six and nine.

We would like to mention that the spaces we discuss in this text are canonical candidates for supergravity backgrounds. For more details on the supergravity point of view that go beyond the explanations in our text we cordially refer the reader to the literature in the references, e.g. [1–6].

2. The general setup

The classification of solvable Lorentzian symmetric spaces by the construction presented below goes back to [7]. Let $(V, \langle\cdot,\cdot\rangle)$ be an n-dimensional euclidean vector space and B be a symmetric endomorphism of V. We denote the symmetric bilinear form that is defined by B and $\langle\cdot,\cdot\rangle$ by the same symbol B and we write $* : V \to V^*, v \mapsto v^*$ with $v^*(w) = \langle v, w\rangle$ for the canonical identification of V and its dual. We define $W := \mathbb{R}^{1,1} \oplus V$ and denote by $\mathring{g}$ the extension of $\langle\cdot,\cdot\rangle$ to a block diagonal Lorentzian metric on W. We consider a null basis $\{e_+, e_-\}$ of $\mathbb{R}^{1,1}$ with respect to $\mathring{g}|_{\mathbb{R}^{1,1}}$. The following skew symmetric multiplication on $\mathfrak{g} := V^* \oplus W$ yields a Lie algebra structure on $\mathfrak{g}$:

$$[e_-, w] = w^*, \tag{1}$$

$$\left[v^*, e_-\right] = Bv, \tag{2}$$

$$\left[v^*, w\right] = -v^*(Bw) \cdot e_+ = -\langle Bv, w\rangle \cdot e_+, \tag{3}$$

for all $w \in V$ and $v^* \in V^*$. The bilinear form $\mathring{g}$ is extended to a bi-invariant metric on $\mathfrak{g}$ by $\mathring{g}(v^*, w^*) := \langle Bv, w\rangle$ if B is non degenerate, see below.

Within $\mathfrak{g}$ the factor V^* acts on W, the bracket of W with itself obeys $[W, W] = V^*$, and $\mathring{g}$ is V^*-invariant. From (1) to (3) we see that the embedding

$$V^* \longrightarrow \mathbb{R}_+ \otimes V \hookrightarrow \mathfrak{so}(W) = \mathfrak{so}(V) \oplus (\mathbb{R}_+ \otimes V) \oplus (\mathbb{R}_- \otimes V) \oplus (\mathbb{R}_+ \otimes \mathbb{R}_-) \tag{4}$$

is given by $v^* \mapsto Bv \wedge e_+$ where $x \wedge y(z) := \langle y, z\rangle x - \langle x, z\rangle y$.

The above data yield a $(D = n + 2)$-dimensional symmetric space M_B with Lorentz metric determined by $\langle\cdot,\cdot\rangle$ and B. The resulting Lorentzian space M_B is indecomposable if and only if the symmetric map B is non-degenerate. This can best be seen from (1) to (3) if we recall that M_B is decomposable if there exists a V^*-invariant subspace $\tilde{W} \subset W$ such that $\mathring{g}|_{\tilde{W}\times\tilde{W}}$ is non-degenerate, see [8,9]. If B admits zero eigenvalues $\mathring{g}$ is degenerate but it remains a metric if V^* is truncated. In this case the resulting manifold decomposes into a product of a lower dimensional Cahen–Wallach space and an euclidean space.[1] This can also be deduced from the coordinate form of the metric, see (11).

3. The metric and the Killing vector fields

3.1. The metric

For the coordinate description of M_B we may use the exponential map and write for $x = x^+e_+ + x^-e_- + \vec{x} \in W$ with $\vec{x} = \sum_i x^i e_i$

$$\mu(x) := \exp\left(x^+e_+\right)\exp\left(x^-e_-\right)\exp\left(\vec{x}\right).$$

This obeys

$$\partial_+\mu = \exp(x^+e_+)e_+\exp(x^-e_-)\exp(\vec{x}) = \mu(x)e_+ \tag{5}$$

$$\partial_i\mu = \exp(x^+e_+)\exp(x^-e_-)\exp(\vec{x})e_i = \mu(x)e_i \tag{6}$$

$$\begin{aligned}\partial_-\mu &= \exp(x^+e_+)\exp(x^-e_-)e_-\exp(\vec{x}) = \mu(x)\exp(-\vec{x})e_-\exp(\vec{x})\\ &= \mu(x)\Big(e_- + \sum_i x^ie_i^* - \tfrac{1}{2}\sum_{ij} B_{ij}x^ix^je_+\Big)\end{aligned} \tag{7}$$

where we use $\exp(\vec{x}) = \prod_i \exp(x^ie_i)$ and

$$\begin{aligned}e_j^*\exp(x^ie_i) &= \exp(x^ie_i)\left(e_j^* - B_{ij}x^ie_+\right)\\ e_-\exp(x^ie_i) &= \exp(x^ie_i)\left(e_- + x^ie_i^* - \tfrac{1}{2}B_{ii}(x^i)^2e_+\right)\end{aligned} \tag{8}$$

with the symmetric matrix (B_{ij}) defined by $B(e_i) = \sum_j B_{ji}e_j$.

[1] If we refer to $\mathring{g}$ as metric we will always assume this truncation.