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Degenerate horizons, Einstein metrics, and Lens space bundles

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ABSTRACT

We present a new infinite class of near-horizon geometries of degenerate horizons, satisfying Einstein's equations for all odd dimensions greater than five. The symmetry and topology of these solutions is compatible with those of black holes. The simplest examples give horizons of spatial topology $S^3 \times S^2$ or the non-trivial S^3-bundle over S^2. More generally, the horizons are Lens space bundles associated to certain principal torus-bundles over Fano Kähler–Einstein manifolds. We also consider the classification problem for Einstein metrics on such Lens space bundles and derive a family which unifies all the known examples (Sasakian and non-Sasakian).

1. Introduction

The classification of higher dimensional stationary black hole solutions to Einstein's equations is mainly motivated by modern studies of quantum gravity, such as string theory and the gauge/gravity dualities. In this context, extremal black holes are of particular interest due to the fact they do not emit Hawking radiation.

For the sake of being generic, as well as simplicity, in this paper we will consider the vacuum Einstein equations $R_{\mu\nu} = \Lambda g_{\mu\nu}$, where we allow for a cosmological constant Λ. In four dimensions the black hole uniqueness theorem provides a (partial) answer to the classification problem. However, in higher dimensions, black hole uniqueness is violated. The only explicit black hole solutions known are the spherical horizon topology Myers–Perry black holes [1] and the black ring metrics which have $S^1 \times S^2$ horizon topology [2,3].

Nevertheless, various general results are known, which help constrain the general classification problem. By generalising Hawking's horizon topology theorem [4] to higher dimensions, Galloway and Schoen [5] have shown that the spatial topology of the horizon must be such that it admits a positive scalar curvature metric (i.e. positive Yamabe type). By generalising Hawking's rigidity theorem [4], it has also been shown that non-extremal stationary rotating black holes must admit at least $\mathbb{R} \times U(1)$ isometry [6] (for partial results pertaining to extremal rotating black holes see [7]). Both of these topology and symmetry constraints become increasingly weak as one increases the dimensions. Furthermore, there is evidence that black hole uniqueness will be violated much more severely as one increases the dimensions [8,9]. It is clear that in the absence of new ideas, the general classification problem is hopelessly out of reach.

One might expect that more progress could be made by restricting to extremal black holes. To some extent this is the case. The event horizon of all known extremal black holes is a degenerate Killing horizon with compact cross-sections. It

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turns out that restricting Einstein's equations for a D-dimensional spacetime to a degenerate horizon, gives a set of geometric equations for the induced metric γ_{AB} on such $(D-2)$-dimensional cross-sections H, which depend only on quantities intrinsic to H. In the case of vacuum gravity one gets the following equation on H

$$R_{AB} = \frac{1}{2}h_A h_B - \nabla_{(A}h_{B)} + \Lambda\gamma_{AB} \tag{1}$$

where h_A is a 1-form on H (the connection on the normal bundle to H), R_{AB} and ∇ are the Ricci tensor and metric connection of γ_{AB}. By studying solutions to this problem of Riemannian geometry on H, one can thus consider the possible horizon geometries independently of the full parent spacetime. One can understand this feature of degenerate horizons in terms of the near-horizon limit. This limit exists for any spacetime containing a degenerate horizon and allows one to define an associated near-horizon geometry, which must also satisfy the full Einstein equations [10,11]. Classifying near-horizon geometries is then equivalent to classifying solutions to (1). Because we are ultimately interested in black holes, we will assume that H are compact manifolds.

It turns out that static near-horizon geometries in all dimensions are trivial [12] and hence we will be discussing non-static near-horizon geometries.[1] Uniqueness of $D = 4$ near-horizon geometries has been proved subject to the assumption of axisymmetry [14–16]. Indeed this turns out to be the key ingredient for extending the black hole uniqueness theorem to cover the extreme Kerr black hole [17]. The classification of $D \geq 5$ $(\Lambda = 0)$ near-horizon geometries has been solved assuming a $U(1)^{D-3}$ rotational isometry [18,19]. The reason for this success ultimately relies on the fact that Einstein's equations for $\Lambda = 0$ are integrable for spacetimes with $\mathbb{R} \times U(1)^{D-3}$ isometry.[2] For $D = 5$ this symmetry assumption is compatible with both asymptotic flatness and Kaluza–Klein asymptotics. For $D > 5$ though this assumption is only compatible with Kaluza–Klein asymptotics. In the context of asymptotically flat (or globally Anti de Sitter (AdS)) black holes one expects a maximal abelian rotational symmetry group given by the Cartan subgroup of $SO(D-1)$, namely $U(1)^r$ where $r \equiv \lfloor \frac{1}{2}(D-1) \rfloor$. Hence for $D > 5$ we have $r < D-3$, so such cases are not contained in the known classification.

Despite the absence of classification results, a number of near-horizon geometries are known which possess no more than r rotational Killing fields. The Myers–Perry black holes provide a family with spherical horizon topology (including Λ) [20]. In even dimensions an infinite class has been constructed (including Λ), which possess cross-sections of the horizon which are S^2-bundles associated to S^1-bundles over Fano Kähler–Einstein manifolds [21]. They depend on one continuous angular momentum parameter as well as an integer specifying the precise topology. The simplest examples are cohomogeneity-1 horizon geometries on $S^2 \times S^2$ and $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$.[3]

In this paper we will provide an analogous construction in odd dimensions. In particular we consider horizon cross-section manifolds which are the total space of the associated S^3-bundles, and more generally Lens space bundles, to certain principal T^2-bundles over Fano Kähler–Einstein manifolds. In fact we have already demonstrated an infinite class of near-horizon geometries in this class, which all turn out to possess Sasakian horizon metrics [24]. In this sequel we present a more general class of solutions which contain, in addition to the Sasakian horizons, more generic non-Sasakian geometries. The solutions depend on two angular momenta parameters and certain integers which specify the precise topology. They may be considered as doubly-spinning versions of the Sasakian horizons (which possess only one independent angular momentum) [24]. The simplest examples give horizon geometries on $S^3 \times S^2$ and $S^3 \tilde{\times} S^2$ (the latter, representing the non-trivial bundle, is not allowed in the Sasakian case [24]). We emphasise that all the examples we present possess a topology and symmetry compatible with the known constraints for black holes, in particular they are of positive Yamabe type [5] and are oriented cobordant to S^{D-2} [10].

It is worth remarking that solutions to Einstein's equation have also been of interest in differential geometry [25]. Although spacetimes correspond to Lorentzian metrics, one can often analytically continue these to complete Riemannian metrics. Indeed, the first example of an inhomogeneous Einstein metric on a compact manifold was found by Page, by taking a certain limit of the Kerr–de Sitter metrics [23], giving a metric on $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$. This construction was generalised to five [26], and higher dimensions [27], by using the Myers–Perry metrics, resulting in an infinite class of inhomogeneous Einstein metrics on $S^3 \times S^2$ and $S^3 \tilde{\times} S^2$, and higher dimensional generalisations.

In fact it turns out that the classification of *Einstein* metrics (i.e. solutions to (1) with $h = 0$ and $\Lambda > 0$) on S^3-bundles and Lens space bundles associated to principal T^2-bundles over Fano Kähler–Einstein manifolds, is an open problem (of ODE type). A number of examples are known in this class, most notably the Sasaki–Einstein manifolds [28], as well as a number of non-Sasaki examples including those mentioned above [26,29,27,30]. We show how the classification problem for such metrics can be reduced to a single sixth order non-linear ODE. This allows us to derive all the known examples in a unified form, revealing previously overlooked non-Sasakian examples. These may be of interest in non-supersymmetric generalisations of the AdS/CFT correspondence.

1 Recall we are discussing only vacuum gravity. If one allows for matter fields, such as Maxwell fields, then non-trivial static near-horizon geometries are possible [13].

2 In fact this structure was not exploited for the $D = 5$ analysis of [16], where the classification problem for $\Lambda \neq 0$ was also considered and reduced to the solution of a single 6th order non-linear ODE.

3 These are analogues of the Einstein metrics on complex line bundles over Fano Kähler–Einstein manifolds constructed in [22] (which generalise Page's Einstein metric [23]).