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Monopoles on the Bryant–Salamon G_2-manifolds

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ABSTRACT

G_2-monopoles are solutions to gauge theoretical equations on noncompact 7-manifolds of G_2 holonomy. We shall study this equation on the 3 Bryant–Salamon manifolds. We construct examples of G_2-monopoles on two of these manifolds, namely the total space of the bundle of anti-self-dual two forms over the $\mathbb{S}^4$ and $\mathbb{CP}^2$. These are the first nontrivial examples of G_2-monopoles.

Associated with each monopole there is a parameter $m \in \mathbb{R}^+$, known as the mass of the monopole. We prove that under a symmetry assumption, for each given $m \in \mathbb{R}^+$ there is a unique monopole with mass m. We also find explicit irreducible G_2-instantons on $\Lambda^2_-(\mathbb{S}^4)$ and on $\Lambda^2_-(\mathbb{CP}^2)$.

The third Bryant–Salamon G_2-metric lives on the spinor bundle over the 3-sphere. In this case we produce a vanishing theorem for monopoles.

1. Introduction

A G_2-manifold is a seven dimensional manifold X^7 equipped with a Riemannian metric whose holonomy lies in G_2. Equivalently this can be encoded in a 3 form ϕ, which determines the metric. The condition that the holonomy is in G_2 then amounts to ϕ being both closed and coclosed. It is standard to denote $\psi = *\phi$ and to refer to a G_2-manifold as the pair (X^7, ϕ). Let G be a compact, semisimple Lie group with Lie algebra $\mathfrak{g}$ and $P \to X$ be a principal G-bundle. Denote the adjoint bundle $P \times_{(Ad,G)} \mathfrak{g}$ by $\mathfrak{g}_P$ and equip it with an Ad-invariant metric.

Definition 1. A pair (A, Φ) consisting on a connection A on P and Higgs Field $\Phi \in \Omega^0(X, \mathfrak{g}_P)$ is said to be a G_2-monopole if

$$F_A \wedge \psi = *\nabla_A \Phi. \tag{1.1}$$

If (A, Φ) is a G_2-monopole with $\nabla_A \Phi = 0$, then $F_A \wedge \psi = 0$ and such connections are known as G_2-instantons (if $\Phi \neq 0$, the holonomy of A must preserve Φ, so A is also reducible). In fact, an integration by parts (or a maximum principle) argument shows that in the compact case these are the unique solutions to Eq. (1.1). So, in order to study solutions of Eq. (1.1) with $\nabla_A \Phi \neq 0$ one must either admit singularities or let X be noncompact. Eq. (1.1) is invariant under the action of the gauge group $\mathcal{G}$ and one is interested in the moduli space of irreducible monopoles on P

$$\mathcal{M}(X, P) = \{(A, \Phi \neq 0) \mid \text{solving (1.1) and } A \text{ irreducible}\}/\mathcal{G}. \tag{1.2}$$

Donaldson and Segal in [1] suggested that these monopoles might be related to coassociative submanifolds of X. These are 4 dimensional and ψ-calibrated submanifolds, in particular they are volume minimizing in their homology class and are the G_2 analogs of special Lagrangian submanifolds in the Calabi–Yau case, respectively. There are conjectural theories

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due to Dominic Joyce [2], which attempt to define an invariant of a G_2-manifold by counting rigid, compact coassociative submanifolds. In fact, it follows from McLean's work [3] that compact coassociative manifold M deforms in a smooth moduli space of dimension $b_2^-(M)$. Hence, these are rigid when $b_2^-(M) = 0$ (e.g. $M = \mathbb{S}^4, \mathbb{CP}^2$) and one could hope to count these. An alternative way to define an enumerative invariant of G_2-manifolds goes by counting monopoles and the idea is that this may be related to a count of coassociative submanifolds. The general expectation is that under some asymptotic regime where the mass (i.e. the asymptotic value of $|\Phi|$) gets very large, the monopoles concentrate along some coassociative cycles whose homology class is determined by the topological type of the bundle P. Such a concentration phenomena is expected to be modeled on $\mathbb{R}^3$ monopoles along the fibers of the normal bundle to the coassociatives. However, other then on $\mathbb{R}^7 = \mathbb{R}^4 \times \mathbb{R}^3$ where dimensional reduction gives examples by lifting monopoles on $\mathbb{R}^3$, no examples of such monopoles are known to exist and is this question of existence which is addressed in this paper. There are also similar theories on noncompact Calabi–Yau's relating solutions to monopole equations to special Lagrangian cycles. The analytic properties of the monopole equations in both these cases are work for the Ph.D. Thesis of the author. See [4] for examples of monopoles on a noncompact Calabi–Yau.

Some notation needs to be introduced in order to state the main Theorem 1 of the paper. If (M, g_M) is an Einstein, self-dual 4 manifold ($M = \mathbb{S}^4, \mathbb{CP}^2$) with positive scalar curvature, Bryant and Salamon in [5], constructed G_2-metrics on $\Lambda^2_-(M)$, i.e. the total space of the bundle of anti-self-dual 2-forms on M. These examples have large symmetry groups, in each case there is a compact Lie group K acting on $\Lambda^2_-(M)$ with cohomogeneity 1. In such a situation there is a notion of K-homogeneous principal G-bundle, i.e. the K-action on $\Lambda^2_-(M)$ lifts to the total space P. Moreover, the Bryant–Salamon manifolds are asymptotically conical and so the G_2-structure is asymptotic to a conical one ϕ_C on the cone $((1, +\infty)_r \times \Sigma, g_C = dr^2 + r^2 g_\Sigma)$ over a Nearly Kähler manifold (Σ^6, g_Σ), Proposition 2. Let $\rho : \Lambda^2_-(M) \to \mathbb{R}$ be the distance to the zero section and $\hat{K} \subset \Lambda^2_-(M)$ a compact set such that there is a diffeomorphism $\varphi : (1, +\infty)_r \times \Sigma \to X \setminus \hat{K}$ with $r \circ \varphi = \rho|_{X \setminus \hat{K}}$ and

$$|\nabla^j (\varphi^* \phi - \phi_C)|_C = O(r^{\nu - j}),$$

for some $\nu < 0$ and all $j \in \mathbb{N}_0$, where $|\cdot|_C$, ∇ denote the norm and covariant derivative in the conical metric g_C. Then, we shall consider a K-homogeneous principal bundle P such that there is an isomorphism $\varphi^* P|_{X \setminus \hat{K}} \cong \pi^* P_\infty$, where P_∞ is a bundle over Σ and $\pi : (1, +\infty) \times \Sigma \to \Sigma$ the projection. Moreover, we shall suppose that $\mathfrak{g}_P$ is equipped with an Ad-invariant inner product h which is modeled on an inner product h_∞ in $\mathfrak{g}_{P_\infty}$. We will use a combination of this with g_C in order to measure the growth rate of sections of $\Lambda^* \otimes \mathfrak{g}_P$. For example, if a denotes a section of $\Lambda^* \otimes \mathfrak{g}_P$ we shall say it has rate $\delta \in \mathbb{R}$ with derivatives if on $X \setminus \hat{K}$, $|\nabla^j \varphi^* a| = O(r^{\delta - j})$ for all $j \in \mathbb{N}_0$; where $|\cdot|$ denotes a combination the norm g_C with h_∞ and ∇ the connection obtained by twisting the Levi Civita connection of g_C with ∇_∞.

Definition 2. Let P be a K-homogeneous principal G-bundle as above. A monopole (A, Φ) on P is said to have finite mass if there is a connection A_∞ on P_∞ such that $|\nabla^j (\varphi^* A - A_\infty)|_C = O(r^{-1-\epsilon-j})$, for some $\epsilon > 0$, all $j \in \mathbb{N}$ and

$$m(A, \Phi) = \lim_{\rho \to \infty} |\Phi|, \tag{1.3}$$

is well defined and finite. In this case $m(A, \Phi) \in \mathbb{R}^+$ is the mass of the monopole.

Let $\mathcal{G}_{\text{inv}}$ denote the K-invariant gauge transformations, the moduli space of finite mass, invariant monopoles on $P \to \Lambda^2_-(M)$ is defined as

$$\mathcal{M}_{\text{inv}}(\Lambda^2_-(M), P) = \{\text{finite mass, } K\text{-invariant } (A, \Phi) \text{ solving (1.1) and } \nabla_A \Phi \neq 0\} / \mathcal{G}_{\text{inv}}. \tag{1.4}$$

The monopole equations used here are inspired by the monopole equations in 3 dimensions. In the Euclidean $\mathbb{R}^3$ and for structure group $SU(2)$, there is a unique mass 1 and spherically symmetric solution; this is known as the BPS monopole [6] and we shall denote it by $(A^{\text{BPS}}, \Phi^{\text{BPS}})$. Moreover, for structure group $\mathbb{S}^1$ there are no smooth solutions, but a singular one known as the Dirac monopole. It will also be the case for the G_2-monopoles studied here that there are Abelian monopoles having singularities at the zero section. These will be constructed in Sections 3.2.1 and 4.2.1 for $\Lambda^2_-(\mathbb{S}^4)$ and $\Lambda^2_-(\mathbb{CP}^2)$ respectively, and will be called Dirac monopoles by analogy. Below the main result is stated and in Remark 1 an intuitive explanation of some technical statements is given.

Theorem 1. *There are compact Lie groups $K = \text{Spin}(5)$ ($K = SU(3)$) acting with cohomogeneity 1 on $\Lambda^2_-(M)$ for $M = \mathbb{S}^4$ (respectively $M = \mathbb{CP}^2$) and K-homogeneous principal $SU(2)$ (respectively $SO(3)$) bundles P, such that the moduli spaces $\mathcal{M}_{\text{inv}}(\Lambda^2_-(M), P)$ are nonempty and the following hold:*

1. *For all $(A, \Phi) \in \mathcal{M}_{\text{inv}}$, $\Phi^{-1}(0)$ is the zero section, and the mass gives a bijection*
   $$m : \mathcal{M}_{\text{inv}}(\Lambda^2_-(M), P) \to \mathbb{R}^+.$$
2. *Let $R > 0$, and $\{(A_\lambda, \Phi_\lambda)\}_{\lambda \in [\Lambda, +\infty)} \in \mathcal{M}_{\text{inv}}(\Lambda^2_-(M), P)$ be a sequence of monopoles with mass λ converging to $+\infty$. Then there is a sequence $\eta(\lambda, R)$ converging to 0 as $\lambda \to +\infty$ such that for all $x \in M$*
   $$\exp_\eta^*(A_\lambda, \eta \Phi_\lambda)|_{\Lambda^2_-(M)_x}$$