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Noncommutative Riemannian geometry on graphs

Shahn Majid *

Queen Mary University of London, School of Mathematics, Mile End Rd, London E1 4NS, UK

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ABSTRACT

We show that arising out of noncommutative geometry is a natural family of *edge Laplacians* on the edges of a graph. The family includes a canonical edge Laplacian associated to the graph, extending the usual graph Laplacian on vertices, and we find its spectrum. We show that for a connected graph its eigenvalues are strictly positive aside from one mandatory zero mode, and include all the vertex degrees. Our edge Laplacian is not the graph Laplacian on the line graph but rather it arises as the noncommutative Laplace-Beltrami operator on differential 1-forms, where we use the language of differential algebras to functorially interpret a graph as providing a 'finite manifold structure' on the set of vertices. We equip any graph with a canonical 'Euclidean metric' and a canonical bimodule connection, and in the case of a Cayley graph we construct a metric compatible connection for the Euclidean metric. We make use of results on bimodule connections on inner calculi on algebras, which we prove, including a general relation between zero curvature and the braid relations.

1. Introduction

A differential algebra is an algebra A equipped with a pair (Ω^1, d), where Ω^1 is an $A-A$-bimodule and $\mathrm{d} : A \to \Omega^1$ is a linear map obeying the Leibniz rule $\mathrm{d}(ab) = (\mathrm{d}a)b + a\mathrm{d}b$ for all $a, b \in A$. It is usually required that the map $A \otimes A \to \Omega^1$ given by $a \otimes b \mapsto a\mathrm{d}b$ is surjective. This notion features in essentially all approaches to noncommutative geometry and has been applied extensively in the case in which A is noncommutative, such as to a Lie theory of quantum groups. The space Ω^1 plays the role of differentials or 1-forms in differential geometry, but because we do not suppose that the left and right module structures on A are equal (i.e. 1-forms may not commute with functions) this notion is fundamentally more general than conventional differential geometry when specialised to commutative algebras. In particular, it is exactly what is needed to provide a notion of differential geometry on finite sets, where the only ordinary differentiable structure on a discrete topology is the zero one. The present work is mathematical but the need for both discrete and noncommutative geometry is increasingly accepted as a requirement for physics at the Planck scale, i.e. for quantum gravity. Readers are referred to [1–4] and references therein for some specific background and other approaches in the finite set case.

Our first result, Theorem 3.1, makes the finite set case more precise. If X is a finite set, it is well known that differential structures on $A = k(X)$, where k is a field, are in one-to-one correspondence with digraphs with vertex set X [5]. We show that this correspondence is in fact functorial. This means that natural constructions for digraphs can be expressed in terms of differential algebra and vice versa; differential algebra constructions can be specialised. Although many constructions in discrete mathematics are loosely motivated by geometric intuition, the precise nature of our correspondence allows

* Tel.: +44 7910521069.
E-mail address: s.majid@qmul.ac.uk.

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systematic transfer of those ideas of classical differential geometry that can be extended to general differential algebras and then specialised. To this end, much of differential geometry, notably metrics $\mathfrak{g} \in \Omega^1 \otimes_A \Omega^1$ and linear connections $\nabla : \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ on bimodules, their curvature R_∇ and geometric torsion T_∇, are all understood at the level of differential algebra and can be specialised to digraphs, which we do. This framework is used in the literature on noncommutative geometry and we give a short account in Sections 2.1 and 2.2, with some further algebraic results in the rest of Section 2 that are needed later. Notably, Theorem 2.1 relates zero curvature to the braid relations for a generalised flip map σ defined by a connection.

Focusing mainly on the symmetric or bidirected case (i.e. we view an undirected graph as directed both ways), Section 3.2 analyses the most general form of metric and bimodule connection. Notably, in Proposition 3.7, we reproduce the general graph Laplacian on vertices as $(\ ,\ )\nabla\mathrm{d}$ in terms of an inverse metric and a bimodule connection, providing a more extended geometric picture than hitherto available. Section 3.3 provides a canonical $\mathfrak{g}, \nabla$ which is nevertheless quite unusual from the point of view of classical geometry in that the generalised flip map σ associated with the connection is the identity. Applying the preceding analysis, Proposition 3.10 finds a natural zero-curvature 'Maurer–Cartan' connection ∇ on any Cayley graph as a member of a class of connections of permutation type studied in Section 3.5, including their torsion T_∇ and curvature R_∇.

Section 4 presents the main result of the paper, a Laplacian on the edges of a graph. In classical Riemannian geometry, Hodge–de Rham theory provides for a Laplace–Beltrami (and a Hodge–Laplace) operator on all degrees of forms, not just functions, and in Section 4.1 we develop this in degree 1 at the level of differential algebras and connections ∇. Section 4.2 then shows how a natural Laplacian emerges on the edges of the graph as the analogue of the Laplace–Beltrami operator on the space of 1-forms on a manifold. Section 4.3 specialises to the canonical connection and Euclidean metric of Section 3.3 and we study this canonical edge Laplacian in detail. We prove (Theorem 4.9) that the eigenvalues of the canonical edge Laplacian are strictly positive for any connected graph, aside from a single zero mode (just as for the usual graph Laplacian on the vertices). We show that its spectrum has one part that is (twice) that of the usual graph Laplacian on vertices and a second part consisting of the integer degrees of every vertex. Our analysis does not exactly show that the edge Laplacian is diagonalisable, but it typically is; a sufficient condition that we prove is that the two parts of the spectrum are disjoint.

Digraph geometry also provides a good illustration of nonclassical ideas in noncommutative geometry, since it is fundamentally noncommutative even though the algebra $k(V)$ is commutative. The most important of these ideas is that sufficiently noncommutative geometries tend to be inner in the sense of a differential 1-form $\theta \in \Omega^1$ such that $[\theta, a] = \mathrm{d}a$ for all $a \in A$ (here $[\theta, a] = \theta a - a\theta$). There can be no such concept in classical differential geometry, as 1-forms and functions commute. However, if classical geometry is a limit of a noncommutative geometry, then there can be phenomena in classical geometry that are unconnected but which become connected or explicable in the noncommutative case. We illustrate this with the Laplacian, which we show (Proposition 4.3) has a deeper origin as the partial derivative conjugate to the direction of θ.

2. General framework

We briefly describe a formulation of noncommutative Riemannian differential algebras that we apply in the rest of the paper. Sections 2.1 and 2.2 are not intended to contain anything new. The other sections contain some new material for Laplacians and geometry with inner calculi, motivated by a previous study [6].

2.1. Differential algebras

Let k be a ground field of characteristic not 2 and let A be a unital algebra over k viewed as coordinate algebra or functions on a space (it can, however, be noncommutative). Throughout the paper, we denote $\bar{\otimes} = \otimes_A$ for brevity. A differential algebra structure on A (or differential structure) means a specification of a space Ω^1 of 1-forms and a linear map d or exterior derivative obeying the following:

(1) Ω^1 is an $A-A$-bimodule (so $f(\omega g) = (f\omega)g$ for all $f, g \in A,\ \omega \in \Omega^1$).
(2) $\mathrm{d} : A \to \Omega^1$ is a derivation $\mathrm{d}(fg) = (\mathrm{d}f)g + f\mathrm{d}g$.
(3) $A \otimes A \to \Omega^1$ by $f \otimes g \mapsto f\mathrm{d}g$ is surjective.
(4) (Optional connectedness condition) $\ker \mathrm{d} = k1$.

A morphism of differential algebras $(A, \Omega^1, \mathrm{d}) \to (B, \Omega^1, \mathrm{d})$ means an algebra map $\phi : A \to B$ and a compatible $\phi_* : \Omega^1(A) \to \Omega^1(B)$ such that

$$\begin{array}{ccc} \Omega^1(A) & \xrightarrow{\phi_*} & \Omega^1(B) \\ \mathrm{d}\nwarrow & & \nearrow\mathrm{d} \\ A & \xrightarrow{\phi} & B \end{array} \tag{2.1}$$

commutes. The surjectivity condition (3) means that ϕ_* if it exists is uniquely determined once ϕ is specified; in other words, this diagram defines the requirement for a map between differentiable algebras to be differentiable.