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First variation formula and conservation laws in several independent discrete variables

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ABSTRACT

This paper sets the scene for discrete variational problems on an abstract cellular complex that serves as discrete model of $\mathbb{R}^p$ and for the discrete theory of partial differential operators that are common in the Calculus of Variations. A central result is the construction of a unique decomposition of certain partial difference operators into two components, one that is a vector bundle morphism and other one that leads to boundary terms. Application of this result to the differential of the discrete Lagrangian leads to unique discrete Euler and momentum forms not depending either on the choice of reference on the base lattice or on the choice of coordinates on the configuration manifold, and satisfying the corresponding discrete first variation formula. This formula leads to discrete Euler equations for critical points and to exact discrete conservation laws for infinitesimal symmetries of the Lagrangian density, with a clear physical interpretation.

1. Introduction

Discrete models in geometry were always present in mathematics both for its intrinsic interest and for the insight they let for more complicated related problems. This area is receiving an increasing interest in modern mathematics ([1–6] and references therein). When combined with modern computational tools, these models open new perspectives for the analysis of problems with physical or geometrical origin that involve ordinary or partial differential equations. An important part of these equations have its origin in a variational principle, with or without constraints [7–15].

In the modern theory of numerical algorithms that model physical problems it is becoming clear [5–7,14–18] the need of a formalization of the variational theory for a discrete Lagrangian. Within this theory one may explore the possibility

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to introduce discrete variational objects and its properties. This allows for a comparison with the corresponding objects that are present in the classical smooth theory. Such a study is not new [17,19], but in the past decades has given rise to new numerical integration procedures of mechanical systems with interesting geometrical and long-term properties [5,7,14,16,18,20–23]. However, for several independent variables, that is, for PDEs of field theories, discrete variational integrators have been studied only in recent years [12,13,16,24,25].

Historically, the first variational problem introduced by Johann Bernoulli (the determination of the brachistochrone curve) was solved by him taking the curve as composed of several rectilinear elements [26]. That is, he studied the problem through an associated discrete model. After a more rigorous formalization of the subject by Euler and later by Lagrange, within the language of differential calculus, the doctrine evolved to its modern state. On the one hand, and thanks to Weierstrass' foundational work, it represents a fruitful branch of functional analysis. On the other hand, following the path marked among others by Lie, Noether and Cartan, there is the possibility to develop a geometrical study of the equations from a Lagrangian or Hamiltonian point of view, with a language that is independent of coordinates. The formalizing work by Lagrange and later Weierstrass gave it a sound theoretical foundation, while Lie's theory of continuous groups led to different key results of ODEs and PDEs theory, for example Noether's conservation laws. Application of Cartan's calculus brought geometrical character to the objects in the theory.

One of the keystones of the calculus of variations is first variation formula. We briefly sketch how it is obtained and its consequences. Consider a function $L(x^\nu, y^j, y^j_\nu)$ (the "Lagrangian") depending on p independent variables x^ν (where $\nu \in \{1, \ldots, p\}$), on n dependent variables y^j (where $j \in \{1, \ldots, n\}$) and on $p \cdot n$ variables y^j_ν. Consider the volume form $\mathrm{vol}_X = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^p$. Consider a compact domain $A \subset \mathbb{R}^p$ with regular boundary ∂A. For any regular mapping $y\colon x = (x^\nu) \in \mathbb{R}^p \mapsto y^j(x) \in \mathbb{R}^n$ we may define the action functional $\mathbb{L}_A(y) = \int_{x\in A} L(x^\nu, y^j(x), \partial y^j(x)/\partial x^\nu)\mathrm{vol}_X \in \mathbb{R}$. For any 1-parameter family of mappings $y_t = (y^j_t(x))$ (seen as a variation of a given mapping y_0) and for the corresponding infinitesimal variation $\delta y = \left((\mathrm{d}/\mathrm{d}t)_{t=0}\, y^j_t(x)\right)$ there holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{L}_A(y_t)\bigg|_{t=0} = \int_{x\in A} \left(\delta y^j \cdot (\partial L/\partial y^j) + \delta y^j_\nu \cdot (\partial L/\partial y^j_\nu)\right)(x^\nu, y^j(x), \partial y^j(x)/\partial x^\nu)\mathrm{vol}_X$$

this integrand contains the term $\delta y^j_\nu = \partial(\delta y^j)/\partial x^\nu$, which involves the derivatives of the infinitesimal variation δy. With a simple integration by parts we obtain first variation formula, where these derivatives disappear:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{L}_A(y_t)\bigg|_{t=0} &= \int_{x\in\partial A} \left((\partial L/\partial y^j_\nu)(x^\nu, y^j(x), \partial y^j(x)/\partial x^\nu)\right) \cdot \delta y^j(x)\mathrm{vol}^\nu_X \\ &\quad + \int_{x\in A}\left((\partial L/\partial y^j)(x^\nu, y^j(x), \partial y^j(x)/\partial x^\nu) - \frac{\mathrm{d}}{\mathrm{d}x^\nu}(\partial L/\partial y^j_\nu)(x^\nu, y^j(x), \partial y^j(x)/\partial x^\nu)\right)\cdot \delta y^j(x)\mathrm{vol}_X\end{aligned}$$

where vol^ν_X stands for the $(p-1)$-form obtained by contraction of $\partial/\partial x^\nu$ with the volume form vol_X. This is first variation formula for the functional $\mathbb{L}_A$. The interested reader can find details and technicalities for a more geometrical presentation of this formula in [8–11]. We may write then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{L}_A(y_t)\bigg|_{t=0} = \int_A \mathrm{d}\left(\langle\omega_y, \delta y\rangle\right) + \langle\mathcal{E}(y)\cdot\mathrm{vol}_X, \delta y\rangle = \int_{\partial A}\langle\omega_y, \delta y\rangle + \int_A \langle\mathcal{E}(y), \delta y\rangle\mathrm{vol}_X$$

where $\mathcal{E}(y)\cdot\mathrm{vol}_X$ is a p-form on X with values on $\mathrm{span}(\mathrm{d}_{y(x)}y^j)_{j=1\ldots n}$, called Euler form associated to the Lagrangian density $L\cdot\mathrm{vol}_X$ and to the mapping y and where ω_y is a $(p-1)$-form on X with values on $\mathrm{span}(\mathrm{d}_{y(x)}y^j)_{j=1\ldots n}$.

For each infinitesimal variation whose support is interior to A we have $\delta y(x) = 0$ on $x \in \partial A$, therefore $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{L}_A(y_t)\big|_{t=0} = \int_A\langle\mathcal{E}(y), \delta y\rangle\mathrm{vol}_X$. Following the main lemma of the calculus of variations, the vanishing of this integral for any such δy is equivalent to Euler–Lagrange equations $\mathcal{E}(y) = 0$ on the interior of A. These equations characterize mappings that are critical for $\mathbb{L}$ among those with a fixed boundary.

If we have a 1-parameter family of symmetries φ_t of the Lagrangian, we may consider a 1-parameter family y_t of mappings obtained applying φ_t to some given mapping and the corresponding infinitesimal variation δy. Then $\mathbb{L}_A(y_t)$ is constant. If $\mathcal{E}(y_0) = 0$, we get $0 = \int_A \mathrm{d}\left(\langle\omega_{y_0}, \delta y\rangle\right)$. This goes to the boundary due to Stokes' Theorem and allows to recover conserved quantities: $0 = \int_{\partial A}\langle\omega_{y_0}, \delta y\rangle$. The "Noether current" $\langle\omega_{y_0}, \delta y\rangle$ vanishes when integrated on ∂A if y_0 is critical. Hence first variation formula leads straightforward to Euler equations and to Noether currents and conservation laws associated to symmetries of the Lagrangian.

The boundary term ω_y is the momentum form. It determines a Legendre transformation and leads to the associated Cartan form of the problem. In this way one may recover the whole Hamilton–Cartan theory, with the characterization of critical mappings by means of the de Donder–Weyl equations, Poisson bracket for field theories, etc. (see [10,11]). It is known that both Cartan and Euler forms can be constructed univocally as tensors from the Lagrangian density, and that this construction is functorial when one considers the category of fibered smooth manifolds and fibered isomorphisms: If one considers a local isomorphism $\varphi\colon (x, y) \mapsto (\bar{x}(x), \bar{y}(x, y))$ and the natural action $(\varphi\cdot)$ of this isomorphism on the corresponding tensor spaces, when one applies $\varphi\cdot$ to Cartan and Euler forms associated to $L\mathrm{vol}_X$ one obtains precisely Cartan and Euler forms associated to $\varphi\cdot(L\mathrm{vol}_X)$.