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Nonholonomic Hamilton–Jacobi theory via Chaplygin Hamiltonization

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ABSTRACT

We develop Hamilton–Jacobi theory for Chaplygin systems, a certain class of nonholonomic mechanical systems with symmetries, using a technique called Hamiltonization, which transforms nonholonomic systems into Hamiltonian systems. We give a geometric account of the Hamiltonization, identify necessary and sufficient conditions for Hamiltonization, and apply the conventional Hamilton–Jacobi theory to the Hamiltonized systems. We show, under a certain sufficient condition for Hamiltonization, that the solutions to the Hamilton–Jacobi equation associated with the Hamiltonized system also solve the nonholonomic Hamilton–Jacobi equation associated with the original Chaplygin system. The results are illustrated through several examples.

1. Introduction

1.1. Background and motivation

In 1911, S.A. Chaplygin published a paper (re-published in English in [1]) introducing his theory of the "reducing multiplier" into the study of nonholonomically constrained mechanical systems. In his paper, Chaplygin showed that a two degree of freedom nonholonomic system possessing an invariant measure became Hamiltonian after a suitable reparameterization of time, a process we would like to refer to as *Chaplygin Hamiltonization*. Since then, Chaplygin's result has generated considerable interest and has been extended [2–6] to more general settings.

However, a second contribution contained in Chaplygin's paper has been left undeveloped. In Section 5 of his paper, Chaplygin integrates the nonholonomic system now known as the Chaplygin Sleigh [7] by using the Hamilton–Jacobi equation for the Hamiltonized system. The aim of this paper is to develop this idea further to establish a link with the nonholonomic Hamilton–Jacobi equation in [8,9].

Specifically, we first employ the technique called Chaplygin Hamiltonization to transform Chaplygin systems into Hamiltonian systems, and then apply the conventional Hamilton–Jacobi theory to the resulting Hamiltonian systems to obtain what we would like to call the *Chaplygin Hamilton–Jacobi equation*. This is an indirect approach towards Hamilton–Jacobi theory for nonholonomic systems, compared to the direct approach of extending Hamilton–Jacobi theory to nonholonomic systems, as in [8,10,9,11].

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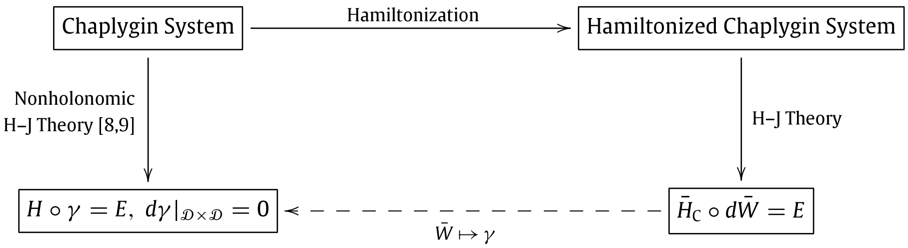

Fig. 1. Relationship between the nonholonomic H–J equation applied to a Chaplygin system and the H–J equation applied to the Hamiltonized Chaplygin system. Explicit formulas for the correspondence $\bar{W} \mapsto \gamma$ are given in Theorems 4.1 and 7.1.

1.2. Direct vs. indirect approaches

The indirect approach to nonholonomic Hamilton–Jacobi theory via Chaplygin Hamiltonization has both advantages and disadvantages. The main advantage is that we have a conventional Hamilton–Jacobi equation and thus the separation of variables argument applies in a rather straightforward manner compared to the direct approach in [9]. A disadvantage is that the Chaplygin Hamiltonization works only for certain nonholonomic systems; even if it does, the relationship between the Hamilton–Jacobi equation and the original nonholonomic system is not transparent, since one has to inverse transform the information in the Hamiltonized systems. Nevertheless, Hamiltonization is known to be a powerful technique for integration of nonholonomic systems [1,4,12,13,6], and hence it is interesting to establish a connection with the direct approach.

Let us briefly summarize the differences between two approaches. Recall from [9] that the *nonholonomic Hamilton–Jacobi equation* is an equation for a one-form γ on the original configuration manifold Q:

$$H \circ \gamma = E, \tag{1.1}$$

along with the condition that γ, seen as a map from Q to T^*Q, takes values in the constrained momentum space $\mathcal{M} \subset T^*Q$ (see Eq. (2.4) below), i.e., $\gamma : Q \to \mathcal{M}$, and also that

$$d\gamma|_{\mathcal{D}\times\mathcal{D}} = 0, \text{ i.e., } d\gamma(v, w) = 0 \quad \text{for any } v, w \in \mathcal{D}, \tag{1.2}$$

where $\mathcal{D} \subset TQ$ is the distribution defined by nonholonomic constraints, and $H : T^*Q \to \mathbb{R}$ the Hamiltonian.

On the other hand, the Chaplygin Hamiltonization first reduces the system by identifying it as a so-called Chaplygin system with a symmetry group G, and then Hamiltonizes the system on the cotangent bundle $T^*(Q/G)$ of the reduced configuration space Q/G. The resulting system is a (strictly) Hamiltonian system on $T^*(Q/G)$ with another Hamiltonian $\bar{H}_C : T^*(Q/G) \to \mathbb{R}$; so we may apply the conventional Hamilton–Jacobi theory to the Hamiltonized system to obtain the *Chaplygin Hamilton–Jacobi equation*

$$\bar{H}_C \circ d\bar{W} = E,$$

which is a partial differential equation for a function $\bar{W} : Q/G \to \mathbb{R}$. Therefore, the difference lies not only in the forms of the equations (the former involves the one-form γ, which is not even closed, whereas the latter invokes the exact one-form $d\bar{W}$), but also in the spaces on which the equations are defined. Furthermore, the Chaplygin Hamilton–Jacobi equation corresponds to the Hamiltonized dynamics and is related to the original nonholonomic one in a rather indirect way. Therefore, on the surface, there does not seem to be an apparent relationship between the two approaches.

1.3. Main results

The main goal of this paper is to establish a link between the two distinct approaches towards Hamilton–Jacobi theory for nonholonomic systems. To that end, we first formulate the Chaplygin Hamiltonization in an intrinsic manner to elucidate the geometry involved in the Hamiltonization. This gives a slight generalization of the Chaplygin Hamiltonization by Fedorov and Jovanović [4] and also an intrinsic account of the necessary and sufficient condition for Hamiltonizing a Chaplygin system presented in [6]. These results are also related to the existence of an invariant measure in nonholonomic systems (see, e.g., [14,15,4]).

We also identify a sufficient condition for the Chaplygin Hamiltonization, which turns out to be identical to one of those for another kind of Hamiltonization (which renders the systems "conformal symplectic" [5]) obtained by Stanchenko [2] and Cantrijn et al. [16]. We then give an explicit formula that transforms the solutions of the Chaplygin Hamilton–Jacobi equation into those of the nonholonomic Hamilton–Jacobi equation (see Fig. 1). Interestingly, it turns out that the sufficient condition plays an important role here as well. We also present an extension of these results to a class of systems that are Hamiltonizable after reduction by two stages, following the idea of Hochgerner and García-Naranjo [5]. We illustrate, through several examples, that the Chaplygin Hamilton–Jacobi equation may be solved by separation of variables, and that the solutions are identical to those obtained by Ohsawa and Bloch [9] after the transformation mentioned above.