Journal of Geometry and Physics 61 (2011) 1378–1384

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

On possible Chern classes of stable bundles on Calabi–Yau threefolds

Björn Andreas [a,*], Gottfried Curio [b]

[a] Institut für Mathematik, Humboldt-Universität zu Berlin, Rudower Chaussee 25, 12489 Berlin, Germany

[b] Arnold-Sommerfeld-Center for Theoretical Physics, Department für Physik, Ludwig-Maximilians-Universität München, Theresienstr. 37, 80333 München, Germany

ARTICLE INFO

Article history:
Received 27 October 2010
Received in revised form 23 February 2011
Accepted 2 March 2011
Available online 10 March 2011

Keywords:
Moduli spaces
Calabi–Yau threefolds
Stable bundles
Heterotic string theory
Anomaly cancellation

ABSTRACT

Supersymmetric heterotic string models, built from a Calabi–Yau threefold X endowed with a stable vector bundle V, usually lead to an anomaly mismatch between $c_2(V)$ and $c_2(X)$; this leads to the question whether the difference can be realized by a further bundle in the hidden sector. In [M.R. Douglas, R. Reinbacher, S.-T. Yau, Branes, Bundles and Attractors: Bogomolov and Beyond, math.AG/0604597], a conjecture is stated which gives sufficient conditions on cohomology classes on X to be realized as the Chern classes of a stable reflexive sheaf V; a weak version of this conjecture predicts the existence of such a V if $c_2(V)$ is of a certain form. In this note, we prove that on elliptically fibered X infinitely many cohomology classes $c \in H^4(X, \mathbf{Z})$ exist which are of this form and for each of them a stable $SU(n)$ vector bundle with $c = c_2(V)$ exists.

1. Introduction

To get $N = 1$ heterotic string models in four dimensions, one compactifies the ten-dimensional heterotic string on a Calabi–Yau threefold X which is furthermore endowed with a polystable holomorphic vector bundle V'. Usually one takes $V' = (V, V_{hid})$ with V, a stable bundle, considered to be embedded in (the visible) E_8 (V_{hid} plays the corresponding role for the second hidden E_8); the commutant of V in E_8 gives the unbroken gauge group in four dimensions.

The most important invariants of V are its Chern classes $c_i(V)$, $i = 0, 1, 2, 3$. We consider in this note bundles with $c_0(V) = rk(V) = n$ and $c_1(V) = 0$; more specifically we will consider $SU(n)$ bundles. The net number of generations of chiral particle multiplets in the four-dimensional effective theory is given by $N_{gen}(V) = |c_3(V)/2|$. On the other hand, the second Chern class is important to ensure anomaly freedom of the construction; this is encoded in the integrability condition for the existence of a solution to the anomaly cancelation equation

$$c_2(X) = c_2(V) + W. \tag{1.1}$$

Here, W, as it stands, has just the meaning to indicate a possible mismatch for a certain bundle V; it can be understood either as the cohomology class of (the compact part of the world volume of) a fivebrane, or as second Chern class of a further stable bundle V_{hid} in the hidden sector. Furthermore, in the first case the class of W has to be effective for supersymmetry to be preserved.

Often in heterotic string compactifications, we are interested in the question whether a stable bundle with a particular $c_3(V_{vis})$ exists, since $c_3(V_{vis})$ determines the number of generations of the heterotic string compactification. Conversely, if we have such a visible bundle, its second Chern class must satisfy Eq. (1.1); thus we are now interested in the question whether a stable hidden bundle exists with a particular $c_2(V_{hid}) = c_2(X) - c_2(V_{vis})$.

* Corresponding author.
E-mail addresses: andreas@math.hu-berlin.de (B. Andreas), gottfried.curio@physik.uni-muenchen.de (G. Curio).

0393-0440/$ – see front matter
doi:10.1016/j.geomphys.2011.03.001

In [1], it has been shown that whenever the topological constraint can be satisfied with $W = 0$, then X and V can be deformed to a solution of the anomaly equation even already on the level of differential forms (a solution to the system involving the three-form field-strength H, investigated first in [2], exists).

This leads to the general question to give sufficient conditions for the existence of stable bundles with prescribed Chern classes $c_2(V)$ and $c_3(V)$. Concerning this, the following conjecture is put forward in [3] by Douglas, Reinbacher and Yau (DRY) (actually we use the particular case of the conjecture with $c_1(V) = 0$).

DRY-Conjecture. *On a Calabi–Yau threefold X of $\pi_1(X) = 0$ a stable reflexive sheaf V of rank n and $c_1(V) = 0$ with prescribed Chern classes $c_2(V)$ and $c_3(V)$ will exist if, for an ample class $H \in H^2(X, \mathbf{R})$, these can be written as (where $C := 16\sqrt{2}/3$)*

$$c_2(V) = n\left(H^2 + \frac{c_2(X)}{24}\right) \tag{1.2}$$

$$c_3(V) < CnH^3. \tag{1.3}$$

Note that the conjecture just predicts the existence of a stable reflexive sheaf; in our examples below V will be a vector bundle.

We will also formulate a weaker version of the conjecture, which is implied by the proper (strong) form and concentrates on the existence of V given that just its (potential) $c_2(V)$ fulfills the relevant condition. To refer more easily to the notions involved, first we give the following definitions. We restrict to the case of V being a vector bundle. We will consider rank n bundles of $c_1(V) = 0$ and treat actually the case of $SU(n)$ vector bundles.

Definition. Let X be a Calabi–Yau threefold of $\pi_1(X) = 0$ and $c \in H^4(X, \mathbf{Z})$,

(i) c is called a *Chern class* if a stable $SU(n)$ vector bundle V on X exists with $c = c_2(V)$
(ii) c is called a *DRY class* if an ample class $H \in H^2(X, \mathbf{R})$ exists (and an integer n) with

$$c_2(V) = n\left(H^2 + \frac{c_2(X)}{24}\right). \tag{1.4}$$

With these definitions we can now state the weak DRY-conjecture, in the framework as we will use it, as follows.

Weak DRY-Conjecture. *On a Calabi–Yau threefold X of $\pi_1(X) = 0$ every DRY class $c \in H^4(X, \mathbf{Z})$ is a Chern class.*

Here it is understood that the integer n occurring in the two definitions is the same.

The paper has three parts. In Section 2, we give criteria for a class to be a DRY class. In Section 3, we present some bundle constructions and show that their $c_2(V)$ fulfill these criteria for infinitely many V.

2. DRY classes on elliptic Calabi–Yau threefolds

To test these conjectures we choose X to be elliptically fibered over the base surface B with section $\sigma : B \longrightarrow X$ (we will also denote by σ the embedded subvariety $\sigma(B) \subset X$ and its cohomology class in $H^2(X, \mathbf{Z})$), a case particularly well studied. The typical examples for B are rational surfaces like a Hirzebruch surface $\mathbf{F_k}$ (where we consider the following cases $k = 0, 1, 2$ as only for these bases exists a smooth elliptic X with Weierstrass model), a del Pezzo surface $\mathbf{dP_k}$ ($k = 0, \ldots, 8$) or the Enriques surface (or suitable blow-ups of these cases). We will consider specifically bases B for which $c_1 := c_1(B)$ is ample. This excludes, in particular, the Enriques surface and the Hirzebruch surface[1] $\mathbf{F_2}$. (The classes c_1^2 and $c_2 := c_2(B)$ will be considered as (integral) numbers.)

On the elliptic Calabi–Yau space X, one has according to the general decomposition $H^4(X, \mathbf{Z}) \cong H^2(X, \mathbf{Z})\sigma \oplus H^4(B, \mathbf{Z})$, the decompositions (with $\phi, \rho \in H^2(X, \mathbf{Z})$)

$$c_2(V) = \phi\sigma + \omega \tag{2.1}$$

$$c_2(X) = 12c_1\sigma + c_2 + 11c_1^2 \tag{2.2}$$

where ω is understood as an integral number (pullbacks from B to X will be usually suppressed).

One now solves for $H = a\sigma + \rho$ (using the decomposition $H^2(X, \mathbf{Z}) \cong \mathbf{Z}\sigma \oplus H^2(B, \mathbf{Z})$), given an arbitrary but fixed class $c = \phi\sigma + \omega \in H^4(X, \mathbf{Z})$, and has then to check that H is ample. The conditions for H to be ample are (cf. Appendix)

$$H \text{ ample} \iff a > 0, \qquad \rho - ac_1 \text{ ample}. \tag{2.3}$$

[1] As $c_1 \cdot b = (2b + 4f) \cdot b = 0$, using here the notations from footnote 6.