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On eigenvalue estimates for the Dirac–Witten-type operators

Yongfa Chen

School of Mathematics and Statistics, Central China Normal University, Wuhan 430079, PR China

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ABSTRACT

We give optimal lower bounds for the eigenvalues of the Dirac–Witten-type operators associated with the e_0-Killing connection and imaginary Killing connection, in terms of the mean curvature and the scalar curvature. The limiting cases are then studied and lead to interesting geometric situations.

1. Introduction

It is well known that the spectrum of the Dirac operator on closed spin manifolds reveals subtle information on the geometry and the topology of such manifolds. In fact, by the Schrödinger–Lichnerowicz formula and the index theorem, there exists a topological obstruction to manifolds supporting positive scalar curvature metrics [1]. Manifolds with minimal eigenvalues are characterized by the existence of solutions of some overdetermined systems such as Killing spinor fields, which imply severe restrictions on the holonomy [2].

For compact spin manifolds with a boundary, such results still hold under certain natural boundary conditions [3].

On the other hand, the well-known spinorial proof of the positive mass theorem given by Witten is based on a subtle use of the Weitzenböck-type formula for a hypersurface Dirac-type operator called the Dirac–Witten operator in [4,5]. The mass is given by the limit at infinity of some boundary integral term. In view of the important role played by the Dirac–Witten operator in Witten's proof, in [6] optimal lower bounds of the Dirac–Witten operator for the dominant energy condition were first given by Hijazi and Zhang in terms of the energy–momentum tensor of the ambient Lorentzian space and the spinorial energy–momentum tensor. Equality cases were also investigated.

When the gravity is involved, it is quite interesting to ask whether the positive mass conjecture still holds if T_{00} is negative on a certain compact set. The question was first studied in [7]. In [8], Zhang proved the positive mass theorem for a class of modified asymptotically flat manifolds in this case by modifying the total linear momentum and the Weitzenböck-type formula. Motivated by [8], we got optimal lower bounds for the eigenvalues of the Dirac–Witten operator of Lorentzian manifolds under certain conditions, in terms of the mean curvature and the scalar curvature and the spinorial energy–momentum tensor [9]. In the limiting case, the spacelike hypersurface is either a maximal and Einstein manifold with positive scalar curvature or the Ricci-flat manifold with nonzero constant mean curvature. Afterwards, by establishing two Weitzenböck formulae for the Dirac–Witten-type operators associated with the e_0-Killing connection and the imaginary Killing connection, Xie [10] proved two hyperbolic positive mass theorems under respective modified energy conditions. Note that in [11], e_0-Killing spinors were used to obtain the corresponding Lorentzian version of the positive mass theorem in [12] and, meanwhile, imaginary Killing spinors were used to obtain another positive mass theorem analogous to the one in [13].

E-mail address: yfchen@amss.ac.cn.

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The present paper is an analogous hyperbolic version of the author's recent work in [9]. In terms of a positive definite Hermitian product on the spinor bundle, we give optimal lower bounds for the eigenvalues of the Dirac–Witten-type operators (see inequalities (3.7) and (4.6)). These estimates are given in terms of the mean curvature and the scalar curvature, and the limiting cases are also discussed. In the last section, as in [3], the previous results still hold in the case of compact spacelike hypersurfaces with non-empty boundary under certain natural local boundary conditions.

2. Preliminaries

We assume that $(N, \tilde{g})$ is an $(n+1)$-dimensional Lorentzian manifold, and $\tilde{g}$ is a Lorentzian metric of signature $(-1, 1, \ldots, 1)$ which satisfies the following Einstein field equations:

$$\widetilde{\mathrm{Ric}} - \frac{1}{2}\tilde{R}\tilde{g} + \Lambda\tilde{g} = T,$$

where $\widetilde{\mathrm{Ric}}$, $\tilde{R}$ are the Ricci curvature and scalar curvature of $\tilde{g}$ respectively, and T is the energy–momentum tensor; $\Lambda = -\frac{n}{\kappa^2}$ is the negative cosmological constant. Let (M, g) be a spacelike spin hypersurface with the induced Riemannian metric g and second fundamental form h. Choose an orthonormal frame $\{e_\alpha\}$ with e_0 timelike. Then in physics, T_{00} is interpreted as the local mass density, and T_{0i} is interpreted as the local momentum density.

Definition 1. A spacelike hypersurface $M \subset N$ satisfies the modified energy condition (1) if, over M,

$$T_{00} \geq |\nabla H| - \frac{H^2}{2n}, \tag{2.1}$$

where $H := \mathrm{tr}_g(h)$.

Definition 2. A spacelike hypersurface M satisfies the modified energy condition (2) if, over M,

$$T_{00} \geq |\nabla H| + \frac{nH}{\kappa} - \frac{H^2}{2n}. \tag{2.2}$$

Note that the weak energy condition says $T_{00} \geq 0$. But modified energy conditions allow that T_{00} can be negative on a certain compact set of M. This fact is of interest when gravity is involved. When M is a maximal spacelike hypersurface, i.e. $H = 0$, then the modified energy conditions turn into the weak energy condition $T_{00} \geq 0$. On the other hand, the modified energy condition (2) goes back to the condition (1) (see [8]) as $\kappa \to \infty$.

Denote by $\mathbb{S}$ the (local) spinor bundle of N. Since M is the spin, $\mathbb{S}$ exists globally over M. This spinor bundle $\mathbb{S}$ is called the hypersurface spinor bundle of M. If $\tilde{\nabla}$ and ∇ denote respectively the Levi-Civita connections of $\tilde{g}$ and g, the same symbols are used to denote their lifts to the hypersurface spinor bundle.

According to [14], there exists a Hermitian inner product (,) on $\mathbb{S}$ over M which is compatible with the spin connection $\tilde{\nabla}$. For any tangent vector $\tilde{X}$ on N, and hypersurface spinors ϕ, ψ, we have

$$(\tilde{X} \cdot \phi, \psi) = (\phi, \tilde{X} \cdot \psi),$$

where "·" denotes the Clifford multiplication. Note that this inner product is not positive definite. Moreover, there exists on $\mathbb{S}$ over M a positive definite Hermitian inner product defined by

$$\langle , \rangle = (e^0 \cdot,).$$

Obviously,

$$\langle e^0 \cdot \phi, \psi \rangle = \langle \phi, e^0 \cdot \psi \rangle$$

and for any tangent vector X on M,

$$\langle X \cdot \phi, \psi \rangle = -\langle \phi, X \cdot \psi \rangle.$$

Fix a point $p \in M$ and an orthonormal basis $\{e_\alpha\}$ with e_0 normal and e_i tangent to M such that $(\nabla_i e_j)_p = 0$ and $(\tilde{\nabla}_0 e_j)_p = 0$. Let $\{e^\alpha\}$ be the dual basis. Then

$$(\tilde{\nabla}_i e^j)_p = -h_{ij} e^0, \qquad (\tilde{\nabla}_i e^0)_p = -h_{ij} e^j,$$

where $h_{ij} = \langle \tilde{\nabla}_i e_0, e_j \rangle$, $1 \leq i, j \leq n$, are the components of the second fundamental form at p. We have the following spinorial Gauss-type formula:

$$\tilde{\nabla}_i = \nabla_i + \frac{1}{2} h_{ij} e^0 \cdot e^j. \tag{2.3}$$