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Transversality of complex linear distributions with spheres, contact forms and Morse type foliations

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ABSTRACT

Let A be a nonsingular $n \times n$ complex matrix. We prove that the corresponding codimension one linear distribution $\mathcal{K}(A)$ is transverse to the unit sphere centered at the origin if and only if $\bar{A}A$ has no positive eigenvalue. If $\mathcal{K}(A)$ is integrable then the corresponding foliation is of Morse type if and only if the eigenvalues of $\bar{A}A$ are pairwise distinct. In this case the variety of contacts with the spheres centered at the origin is a union of n complex lines. We also give a canonical normal form similar to Takagi's unitary normal form. We shall show that the tangency property is robust for small perturbations of the distribution. Finally, we will give a detailed study of an example of a non-Morse type non-linear holomorphic foliation given by the Pham polynomial.

1. Introduction

In the qualitative theory of smooth foliations of codimension one, one main tool is the study of existence of compact transverse sections to the foliation. For instance, classical Haefliger's theorem [1] states that if there is a null-homotopic transverse circle then the foliation exhibits some limit cycle contained in a leaf. In particular, the foliation is not real analytic. The proof strongly relies on the study of vector fields transverse to the boundary of the 2-disk and having Morse type singularities. Since, by topological reasons, there are more centers than saddles in the singular set the use of the Poincaré–Bendixson theorem then shows the existence of a compact invariant divisor (periodic orbit or graph) with one-sided holonomy. These ideas, enforced by the notion of vanishing cycle, are on the basis of the proof of the celebrated Novikov theorem [2] which states that a codimension one smooth foliation on a simply connected three manifold exhibits an invariant solid torus, with boundary leaf diffeomorphic to the torus $S^1 \times S^1$ and interior leaves diffeomorphic to the plane $\mathbb{R}^2$. One very basic common tool in the above results is the use of the Poincaré–Bendixson theorem for planar real vector fields. In the case of holomorphic vector fields, A. Douady and the first named author discovered a Poincaré–Bendixson type theorem [3]. Following this line of research we have for holomorphic foliations of codimension one, the following question [4,5]:

Question (*Main Problem*)**.** *Is there any codimension one holomorphic foliation $\mathcal{F}$ in a neighborhood of the closed unit disk $\overline{D^{2n}(1)} \subset \mathbb{C}^n$ such that $\mathcal{F}$ is transverse to the boundary sphere $S^{2n-1}(1)$ for $n \geq 3$?*

We remark that for $n = 2$ there are linear examples and the situation is well understood [6,3].

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In [4] it is shown that *a linear foliation* $\mathcal{F}$ on $\mathbb{C}^n$, $n \geq 3$, *is not transverse to the sphere* $S^{2n-1}(1)$. *Moreover,* $\mathcal{F}$ *is transverse to the sphere* $S^{2n-1}(1)$ *off the singular set* $\mathrm{Sing}(\mathcal{F}) \cap S^{2n-1}(1)$ *if and only if* $\mathcal{F}$ *is a product* $\mathcal{L}_\lambda \times \mathbb{C}^{n-2}$ *for some linear foliation* $\mathcal{L}_\lambda\colon x\,\mathrm{d}y - \lambda y\,\mathrm{d}x = 0$, *in the Poincaré domain on* $\mathbb{C}^2$.

In [7], we introduced a class of Morse type holomorphic foliations and gave a negative answer to the above question also in the case of Morse type foliations.

As for linear distributions in [7] it is given a sufficient condition for a linear holomorphic foliation on $\mathbb{C}^n$, $n \geq 3$, to be of Morse type. In this paper we first show that it is also a necessary condition. Let $A = (a_{ij})_{i,j}$ be a $n \times n$ complex matrix with $\det A \neq 0$. If $n \geq 3$, the complex linear one-form $\omega_A = \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij} z_j\right) \mathrm{d}z_i$ on $\mathbb{C}^n$ is integrable if and only if A is symmetric; and if it is, all eigenvalues of $\bar{A}A$ are real and positive. We denote by $\mathcal{F}(\omega_A)$ the foliation defined by ω_A and by $\Sigma(\omega_A, \varphi)$ the *variety of contacts* of the foliation $\mathcal{F}(\omega_A)$ and the real codimension one foliation $\mathrm{d}\varphi = 0$, by spheres centered at the origin, where $\varphi = \sum_{j=1}^n |z_j|^2$ is the distance function to the origin.

Theorem 1. *Suppose that* ω_A *is an integrable linear one-form in* $\mathbb{C}^n$. *The foliation* $\mathcal{F}(\omega_A)$ *is of Morse type if and only if the eigenvalues of* $\bar{A}A$ *are pairwise distinct. In this case the variety of contacts* $\Sigma(\omega_A, \varphi)$, *is the union of the n lines given by the eigenvectors of* $\bar{A}A$.

Secondly, if ω_A is not integrable, we will give a necessary and sufficient condition of transversality between $\mathcal{K}(\omega_A)$ and the unit sphere $S^{2n-1}(1) \subset \mathbb{C}^n$, $n \geq 3$.

Theorem 2. *Let* ω_A *be a linear one-form in* $\mathbb{C}^n$, $n \geq 3$. *The distribution* $\mathcal{K}(\omega_A)$ *is transverse to* $S^{2n-1}(1) \subset \mathbb{C}^n$ *if and only if* $\bar{A}A$ *has no positive eigenvalues.*

It is shown in [4,5] that a holomorphic distribution in the odd dimensional space $\mathbb{C}^{2n+1}$ is not transverse to the sphere $S^{4n+1}(1)$ by using the fact that $S^{4n+1}(1)$ has no 2-field. If we restrict ourselves to linear distributions, it is shown that the above theorem also claims the tangency because any matrix $\bar{A}A$ with $\det A \neq 0$ has a positive eigenvalue if n is odd.

In [5], it is proved that a linear skew symmetric distribution is transverse to the sphere $S^{4n-1}(1)$ and homotopic to a *canonical contact* distribution

$$\omega_{(\lambda_1,\ldots,\lambda_n)} := \sum_{j=1}^n \lambda_j (z_{2j}\mathrm{d}z_{2j-1} - z_{2j-1}\mathrm{d}z_{2j})$$

in $\mathbb{C}^{2n}$, $n \geq 2$. We shall show here that *a skew symmetric one-form is in fact equivalent by a linear unitary transformation to a canonical contact form* (Proposition 7). This is done by applying the idea of Takagi's factorization theorem [8,9].

The transversality property is clearly robust under a small perturbation. We shall show that tangency is also robust in the following sense:

Theorem 3. *A linear distribution* $\mathcal{K}(\omega_A)$ *with* $\omega_A = \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij} z_j\right) \mathrm{d}z_i$ *is not transverse to the sphere* $S^{2n-1}(1)$ *provided that it is close enough to a linear foliation of Morse type. If this is the case, let* $f_i = f_i(z_1, \ldots, z_n)$, $i = 1, \ldots, n$, *be holomorphic functions defined in a neighborhood of the origin, such that* $f_i(0) = 0$, $\frac{\partial}{\partial z_j} f_i(0) = 0$, $i, j = 1, \ldots, n$. *Then the holomorphic distribution* $\mathcal{K}(\tilde{\omega})$ *for* $\tilde{\omega} = \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij} z_j + f_i\right) \mathrm{d}z_i$ *is not transverse to the spheres* $S^{2n-1}(r)$ *for small* $r > 0$.

Finally, we will give a detailed study of an example of non-Morse type non-linear holomorphic foliation $\mathcal{F}(\mathrm{d}g)$ defined by the *Pham polynomial* $g(z) = z_1^2 + z_2^3 + z_3^5$ on $\mathbb{C}^3$. We denote by $\varphi(z) = \sum_{j=1}^3 |z_j|^2$ the canonical distance function from the origin in $\mathbb{C}^3$. The variety Σ of contacts between $\mathcal{F}(\mathrm{d}g)$ and $\mathcal{F}(\mathrm{d}\varphi)$ consists of several connected components and we shall prove that

Theorem 4. *The Pham polynomial foliation* $\mathcal{F}(\mathrm{d}g)$ *is of Morse type in the disk* $D^6\left(\frac{2}{3}\right)$ *but not in* $\mathbb{C}^3$. *Indeed the set* Λ *of degenerate points of* Σ *consists of three disjoint circles avoiding the disk* $D^6\left(\frac{2}{3}\right)$.

2. Notations, definitions and reviews

In this section, we prepare some notations and definitions and recall some previous results. Our framework includes non-integrable distributions, *i.e.*, distributions which are not tangent to any codimension one holomorphic foliation. Let $\omega = \sum_{j=1}^n f_j(z)\mathrm{d}z_j$ be a holomorphic one-form defined in an open subset $U \subset \mathbb{C}^n$, $n \geq 2$. The corresponding distribution $\mathcal{K}(\omega)$ in U is defined at each point $p \in U$ by $\mathcal{K}(\omega)(p) = \left\{V \in T_p\mathbb{C}^n;\ \omega(p) \cdot V = 0\right\}$. If $\omega(p) \neq 0$ then $\mathcal{K}(\omega)(p)$ is a $(n-1)$-dimensional linear subspace, while if $\omega(p) = 0$ then $\mathcal{K}(\omega)(p) = T_p\mathbb{C}^n$ and we shall say that the distribution $\mathcal{K}(\omega)$ is singular at p. The singular set of $\mathcal{K}(\omega)$ will be denoted by $\mathrm{Sing}(\mathcal{K}(\omega))$ and coincides with the singular set $\mathrm{Sing}(\omega)$ of ω.

Including the non-integrable case we have the following definition of transversality.