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The geometry of generating functions for a class of Hamiltonians in the noncompact case

P. Guiotto [a], L. Zanelli [b,*]

[a] Dipartimento di Matematica Pura ed Applicata, Università di Padova, via Trieste 63, 35121 Padova, Italy

[b] Dipartimento di Matematica, Università di Bologna, piazza di Porta San Donato 5, 40126 Bologna, Italy

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ABSTRACT

We consider a class of Hamiltonians $H : T^\star\mathbb{R}^n \longrightarrow \mathbb{R}$ and the related flows $\phi_H^t : T^\star\mathbb{R}^n \longrightarrow T^\star\mathbb{R}^n$, proving the existence and uniqueness of generating functions quadratic at infinity for its graph $\Lambda_t = T^*\mathbb{R}^n \times \phi_H^t(T^*\mathbb{R}^n)$. As a consequence, we obtain the same results for the Lagrangian submanifolds $L_t := \phi_H^t(L_0) \subset T^\star\mathbb{R}^n$ Hamiltonianly isotopic to the zero section $L_0 \simeq \mathbb{R}^n$. This problem was also considered by Chaperon, Sikorav and Viterbo in the case of closed manifolds. The assumption on the class of Hamiltonians is an asymptotic behaviour of polynomial type on the phase space. In particular, we deal with a family of Hamiltonian systems arising from usual mechanical problems, for which we study the structure of the corresponding generating functions, showing their main analytical properties. The results presented in the paper are applied to prove the existence and uniqueness of minmax solutions for a class of Hamilton–Jacobi equations on $T^\star\mathbb{R}^n$.

1. Introduction

The problem of existence and uniqueness of generating functions quadratic at infinity for Lagrangian submanifolds has been studied in several papers. The first existence result has been proved by Sikorav in [1] (see also Brunella [2]) for Lagrangian submanifolds obtained from Hamiltonian isotopies on $T^\star M$ in the case of closed manifolds M. In the same setting, the uniqueness has been proved by Viterbo in [3] (see also Thèret [4]). These results are based on the method of broken geodesics introduced by Chaperon [5].

The relevance of these arguments is in several frameworks such as symplectic geometry, Hamiltonian mechanics, geometrical optics and control theory of static systems (see for example [6–9]). The determination of geometrical and minmax solutions for the Hamilton–Jacobi equations is due to Chaperon [10] by using generating functions quadratic at infinity (see also [11–13]). Moreover, they can be used also in the study Hamiltonian optics and thermostatic systems as shown in [6], otherwise for the objective to determine invariants (like capacities) of Lagrangian submanifolds as in [3].

The focus of our work is generating functions quadratic at infinity of Lagrangian submanifolds related to Hamiltonian flows on $T^\star\mathbb{R}^n$. In order to formally state the problem, let us consider $H \in C^2(T^\star\mathbb{R}^n; \mathbb{R})$, the Hamiltonian flow $\phi_H^t : T^\star\mathbb{R}^n \to T^\star\mathbb{R}^n$ and its graph:

$$\Lambda_t := \left\{(x, p; y, \xi) \in T^\star\mathbb{R}^n \times T^\star\mathbb{R}^n \mid (x, p) = \phi_H^t(y, \xi)\right\}.$$

* Corresponding author.
E-mail addresses: parsifal@math.unipd.it (P. Guiotto), zanelli@dm.unibo.it (L. Zanelli).

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We recall that $S : [0, T] \times \mathbb{R}^{2n} \times \mathbb{R}^k \longrightarrow \mathbb{R}$ is a generating function for Λ_t if

$$\Lambda_t = \left\{(x, p; y, \xi) \in T^\star\mathbb{R}^n \times T^\star\mathbb{R}^n \mid p = \partial_x S;\ \xi = -\partial_y S;\ 0 = \partial_\theta S\right\},$$

providing that $0 \in \mathbb{R}^k$ is a regular value of the map $(x, y, \theta) \longmapsto \partial_\theta S(x, y, \theta)$.

Our main result is the existence and uniqueness of generating functions quadratic at infinity and weakly quadratic (in the sense of Thèret [4]) for the Lagrangian submanifolds Λ_t. The Hamiltonians are of the following form:

$$H(x, p) = \frac{1}{2}\langle Gp, p\rangle + \langle a, p\rangle + \langle b, x\rangle + \frac{1}{2}\langle Bx, x\rangle + H_0(x, p), \quad (x, p) \in T^\star\mathbb{R}^n, \tag{1}$$

where G is symmetric and nondegenerate, B is symmetric and $H_0 \in C^2(T^\star\mathbb{R}^n; \mathbb{R})$ is vanishing as $|x| \longrightarrow +\infty$ uniformly in p belonging to bounded sets. Precisely, there exists some $M > 0$, $\beta \geq 0$, $\alpha > \beta + 1$ such that

$$|\nabla^j H_0(x, p)| \leq M \frac{1 + |p|^\beta}{1 + |x|^\alpha}, \quad \forall (x, p) \in T^\star\mathbb{R}^n,\ j = 0, 1, 2. \tag{2}$$

We can think to the previous Hamiltonians H asymptotically (in the x variable) like a second order polynomial. By the Hamilton flow composition rule, we transfer the nonlinear part of the Hamiltonian dynamics from $T^\star\mathbb{R}^n$ to $T^\star\mathbb{S}^n$, via suitable generalizations of the stereographic projection. This idea was implemented by Viterbo [3] in the case of compactly supported symplectic diffeomorphisms and makes possible to apply the known results on existence and uniqueness of generating functions in the closed manifold case. We also remark that the Hamiltonian vector field $X_H = J\nabla H$ may not be globally Lipschitz on $T^\star\mathbb{R}^n$, a general setting not allowing the direct application of the broken geodesics method [5] or the Amann–Conley–Zhender reduction [14].

Thanks to the main result, we can study a family of Lagrangian submanifolds $L_t := \phi_H^t(L_0) \subset T^\star\mathbb{R}^n$ isotopic to the zero section L_0, with generating function $S : [0, T] \times \mathbb{R}^n \times \mathbb{R}^\kappa \longrightarrow \mathbb{R}$.

$$L_t = \left\{(x, p) \in T^\star\mathbb{R}^n \mid p = \partial_x S;\ 0 = \partial_\omega S\right\}.$$

Moreover, taking the Hamiltonians:

$$H(x, p) = \frac{1}{2}\langle Gp, p\rangle + \langle a, p\rangle + \langle b, x\rangle + V(x), \quad (x, p) \in T^\star\mathbb{R}^n, \tag{3}$$

where V satisfy a slightly weak condition of (2), we exhibit a direct construction of a generating function quadratic at infinity for Λ_t. This turns out to be of the form

$$\mathcal{S}(t, x, y, \theta) = \langle M(t)x, x\rangle + \langle M(t)y, y\rangle + \langle \mu(t), x\rangle + \langle \nu(t), y\rangle + \langle w(t, x, y), \theta\rangle + \langle R(t)\theta, \theta\rangle + \mathfrak{S}(t, x, y, \theta) \tag{4}$$

where $\mathfrak{S} \in C_b^1$. Moreover, many analytical features of $\mathcal{S}$ are essentially related to the asymptotic polynomial behaviour of the Hamiltonian.

One important application of our results is to the Cauchy problem for a class of evolutive *Hamilton–Jacobi equations* on $T^\star\mathbb{R}^n$. Shortly, given a Hamiltonian (3), we consider:

$$\begin{cases} \partial_t S(t, x) + H(x, \nabla_x S(t, x)) = 0, & (t, x) \in [0, T] \times \mathbb{R}^n, \\ S(0, x) = \sigma(x), & x \in \mathbb{R}^n. \end{cases} \tag{5}$$

We prove, with the use of the well known theory of minmax critical values, the existence and uniqueness of the minmax solution:

$$S(t, x) = \text{minmax}\, S(t, x, \cdot)$$

where $S(t, x, \cdot)$ is an arbitrary generating function quadratic at infinity for the Lagrangian submanifold $L_t = \phi_H^t\{(x, p) \in T^\star\mathbb{R}^n \mid p = \nabla_x\sigma(x), x \in \mathbb{R}^n\}$. This extend a result of Bernardi and Cardin [13], proved in the case of a mechanical Hamiltonian $H = \frac{1}{2}p^2 + V(x)$ with V compactly supported.

Another application concern the case of Hamiltonians $H(x, p) = \frac{1}{2}\langle G(x)p, p\rangle$ where G is a semi-Riemannian metric asymptotic to the Lorentz metric at infinity (in a suitable sense, the details are discussed in the Remark 6.3). In this framework, our construction leads to a global World Function in a noncompact setting (see also [15]). We note that the present work has a direct application to the Schrödinger equation on $\mathbb{R}^n$, a link already shown in the paper [16] for compact supported potentials.

The content of our paper is the following. Section 2 contains some preliminaries of symplectic geometry. In Section 3 we construct generating functions globally in time in the case of polynomials Hamiltonian. In Section 4, we exhibit generalizations of stereographic projection. These mappings are used in Sections 5 and 6 which is the core of our paper, to prove existence and uniqueness of the generating functions for the class of Hamiltonians satisfying conditions (1). In Section 7 we focus the attention to the case of mechanical Hamiltonians (3) for which we exhibits the structure of a quadratic generating function, while in Section 8 we focus our attention on the determination of global in time solutions for a class of Hamilton–Jacobi equations on $T^\star\mathbb{R}^n$.

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