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On the integrability of symplectic Monge–Ampère equations

B. Doubrov [a,*], E.V. Ferapontov [b]

[a] Department of Mathematical Physics, Faculty of Applied Mathematics, Belarussian State University, Nezavisimosti av. 4, 220030 Minsk, Belarus

[b] Department of Mathematical Sciences, Loughborough University, Loughborough, Leicestershire LE11 3TU, United Kingdom

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This paper is dedicated to Professor Boris Dubrovin on the occasion of his 60th birthday.

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ABSTRACT

Let u be a function of n independent variables $x^1, \dots, x^n$, and let $U = (u_{ij})$ be the Hessian matrix of u. The symplectic Monge–Ampère equation is defined as a linear relation among all possible minors of U. Particular examples include the equation $\det U = 1$ governing improper affine spheres and the so-called heavenly equation, $u_{13}u_{24} - u_{23}u_{14} = 1$, describing self-dual Ricci-flat 4-manifolds. In this paper we classify integrable symplectic Monge–Ampère equations in four dimensions (for $n = 3$ the integrability of such equations is known to be equivalent to their linearisability). This problem can be reformulated geometrically as the classification of 'maximally singular' hyperplane sections of the Plücker embedding of the Lagrangian Grassmannian. We formulate a conjecture that any integrable equation of the form $F(u_{ij}) = 0$ in more than three dimensions is necessarily of the symplectic Monge–Ampère type.

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1. Introduction

Let us consider a function $u(x^1, \dots, x^n)$ of n independent variables and introduce the $n \times n$ Hessian matrix $U = (u_{ij})$ of its second-order partial derivatives. The symplectic Monge–Ampère equation is a partial differential equation (PDE) of the form

$$M_n + M_{n-1} + \cdots + M_1 + M_0 = 0, \tag{1}$$

where M_l is a constant-coefficient linear combination of all $l \times l$ minors of the matrix U, $0 \le l \le n$. Thus, $M_n = \det U = \text{Hess}\, u$, M_0 is a constant, etc. Equivalently, these PDEs can be obtained by equating to zero a constant-coefficient n-form in the $2n$ variables x^i, u_i. Equations of this type belong to the class of completely exceptional Monge–Ampère equations introduced in [1]. Geometric and algebraic aspects of symplectic Monge–Ampère equations have been thoroughly investigated in [2,3]. We point out that the left-hand side of (1), $M(U)$, constitutes the general form of null Lagrangian densities, that is, functionals of the form $\int M(U)\mathrm{d}\mathbf{x}$ which generate trivial Euler–Lagrange equations [4]. The class of Eq. (1) is invariant under

* Corresponding author.
E-mail addresses: doubrov@islc.org (B. Doubrov), E.V.Ferapontov@lboro.ac.uk (E.V. Ferapontov).

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the natural contact action of the symplectic group $Sp(2n)$, which is thus the equivalence group of our problem. All subsequent classification results will be formulated modulo this $Sp(2n)$-equivalence.

When $n = 2$, one arrives at the standard Monge–Ampère equations,

$$\epsilon(u_{11}u_{22} - u_{12}^2) + \alpha u_{11} + \beta u_{12} + \gamma u_{22} + \delta = 0; \tag{2}$$

these are known to be the only equations of the form $F(u_{11}, u_{12}, u_{22}) = 0$ which are linearisable by a transformation from the equivalence group $Sp(4)$.

The case $n = 3$ is also understood completely: one can show that any non-degenerate symplectic Monge–Ampère equation is either linearisable, or $Sp(6)$-equivalent to either of the canonical forms,

$$\text{Hess}\, u = 1, \qquad \text{Hess}\, u = u_{11} + u_{22} + u_{33}, \qquad \text{Hess}\, u = u_{11} + u_{22} - u_{33}; \tag{3}$$

see [2,3,5]; we point out that all three canonical forms are $Sp(6)$-equivalent over $\mathbb{C}$. The first equation arises in the theory of improper affine spheres, while the second describes special Lagrangian 3-folds in $\mathbb{C}^3$ [6,7]. Here the non-degeneracy is understood as follows. Let $F(u_{ij}) = 0$ be a symplectic Monge–Ampère equation. Consider the linearised equation, $\partial F/\partial u_{ij}\, v_{ij} = 0$, obtained by setting $u \to u + \epsilon v$ and keeping terms of the order ϵ. The non-degeneracy means that the corresponding symbol, $\partial F/\partial u_{ij}\, \xi_i\xi_j$, defines an irreducible quadratic form.

The problem of integrability of symplectic Monge–Ampère equations was addressed in [8] based on the method of hydrodynamic reductions [9–11]. Without going into technical details of this method (see the Appendix for a brief summary), let us formulate the main result needed for our purposes.

Theorem 1 (*[8]*). *A non-degenerate three-dimensional symplectic Monge–Ampère equation is integrable by the method of hydrodynamic reductions if and only if it is linearisable.*

In particular, the PDEs (3) are *not* integrable. Although this result is essentially negative, it will be crucial for the classification of integrable equations in higher dimensions (where the situation is far more interesting). Before we proceed to the description of the main results, let us clarify the geometry behind the symplectic Monge–Ampère equations (1) and the linearisability/integrability conditions. Let us consider the Lagrangian Grassmannian Λ, which can be (locally) identified with the space of $n \times n$ symmetric matrices U. The minors of U define the Plücker embedding of Λ into projective space P^N (we identify Λ with the image of its projective embedding). Thus, symplectic Monge–Ampère equations correspond to hyperplane sections of Λ. For $n = 3$, we have $\Lambda^6 \subset P^{13}$, and linearisable equations correspond to hyperplanes which are tangential to Λ^6. Therefore, for $n = 3$, the linearisability condition coincides with the equation of the dual variety of Λ^6, which is known to be a hypersurface of degree 4 in $(P^{13})^*$ (we refer to [12–14] for a general theory behind this example). In Section 2 we provide the following characterisation of linearisable equations in any dimension.

Theorem 2. *For a non-degenerate symplectic Monge–Ampère equation (1) the following conditions are equivalent:*

(1) *The equation is linearisable by a transformation from $Sp(2n)$.*
(2) *The equation is invariant under an n^2-dimensional subalgebra of $Sp(2n)$.*
(3) *The equation corresponds to a hyperplane which contains an osculating subspace O_{n-2} of the Lagrangian Grassmannian of the order $n - 2$.*

For $n = 3$, the third condition reduces to the requirement of tangency.

In Section 3, we address the problem of integrability of symplectic Monge–Ampère equations in four dimensions, $n = 4$. Among the best-known four-dimensional integrable examples one should primarily mention the 'heavenly' equation [15],

$$u_{13}u_{24} - u_{23}u_{14} = 1, \tag{4}$$

which is descriptive of Ricci-flat self-dual 4-manifolds [16]. It was demonstrated in [10] that this equation is integrable by the method of hydrodynamic reductions. Although, in principle, the method of hydrodynamic reductions can be applied in any dimension, it leads to a quite complicated analysis. One way to bypass lengthy calculations is based on the following simple idea. Suppose that our aim is the classification of four-dimensional integrable equations of the form (1) for a function $u(x^1, x^2, x^3, x^4)$. Let us look for travelling wave solutions in the form

$$u = u(x^1 + \alpha x^4, x^2 + \beta x^4, x^3 + \gamma x^4),$$

or, more generally,

$$u = u(x^1 + \alpha x^4, x^2 + \beta x^4, x^3 + \gamma x^4) + Q(x, x),$$

where Q is an arbitrary quadratic form in the variables x^1, x^2, x^3, x^4. The substitution of this ansatz into (1) leads to a three-dimensional symplectic Monge–Ampère equation which must be integrable for *any* values of constants α, β, γ, and an arbitrary quadratic form Q. Since, in three dimensions, the integrability conditions are explicitly known (and are equivalent to the linearisability), this provides strong restrictions on the original four-dimensional equation which are therefore *necessary* for the integrability. In fact, in the present context they turn out to be sufficient: if all three-dimensional equations obtained from a given four-dimensional PDE by travelling wave reductions are linearisable, then the PDE is integrable. The

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