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Almost Einstein and Poincaré–Einstein manifolds in Riemannian signature

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ABSTRACT

An almost Einstein manifold satisfies equations which are a slight weakening of the Einstein equations; Einstein metrics, Poincaré–Einstein metrics, and compactifications of certain Ricci-flat asymptotically locally Euclidean structures are special cases. The governing equation is a conformally invariant overdetermined PDE on a function. Away from the zeros of this function the almost Einstein structure is Einstein, while the zero set gives a scale singularity set which may be viewed as a conformal infinity for the Einstein metric. In this article there are two main results: we give a simple classification of the possible scale singularity spaces of almost Einstein manifolds; we derive geometric results which explicitly relate the intrinsic (conformal) geometry of the conformal infinity to the conformal structure of the ambient almost Einstein manifold. The latter includes new results for Poincaré–Einstein manifolds. Classes of examples are constructed. A compatible generalisation of the constant scalar curvature condition is also developed. This includes almost Einstein as a special case, and when its curvature is suitably negative, is closely linked to the notion of an asymptotically hyperbolic structure.

1. Introduction

A metric is said to be Einstein if its Ricci curvature is proportional to the metric [1]. Despite a long history of intense interest in the Einstein equations many mysteries remain. In high dimensions it is not known if there are any obstructions to the existence of an Einstein metric. There are 3-manifolds and 4-manifolds which do not admit Einstein metrics and the situation is especially delicate in the latter case; see [2] for an overview of some recent progress. Here we consider a specific weakening of the Einstein condition. By its nature this provides an alternative route to studying Einstein metrics but, beyond this, there are several points which indicate that it may be a useful structure in its own right. On the one hand the weakening is very slight, in a sense that will soon be clear. On the other hand, it allows in some interesting cases: at least some manifolds satisfying these equations do not admit Einstein metrics, which suggests a role as a uniformisation type condition; it includes in a natural way Poincaré–Einstein structures and conformally compact Ricci-flat asymptotically locally Euclidean (ALE) spaces, and so Einstein metrics, Poincaré–Einstein structures and these ALE manifolds are special cases of a uniform generalising structure.

Throughout the paper, we consider only metrics g of *Riemannian* signature (meaning that g is positive definite) and the conformal structures these induce; all manifolds shall be assumed to be of dimension $d \geq 3$. On a Riemannian manifold (M^d, g) the Schouten tensor P (or P^g) is a trace adjustment of the Ricci tensor given by

$$\mathrm{Ric}^g = (d-2)P^g + J^g g$$

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where J^g is the metric trace of P^g. Thus a metric is Einstein if and only if the trace-free part of P^g is zero. We will say that (M, g, s) is a *directed almost Einstein* structure if $s \in C^\infty(M)$ is a non-trivial solution to the equation

$$A(g, s) = 0 \quad \text{where } A(g, s) := \text{trace-free}(\nabla^g \nabla^g s + sP^g). \tag{1.1}$$

Here ∇^g is the Levi-Civita connection for g, and the "trace-free" means the trace-free part with respect to taking a metric trace. This is a generalisation of the Einstein condition; we will see shortly that, on the open set where s is non-vanishing, $g^o := s^{-2}g$ is Einstein. On the other hand if g is Einstein then (1.1) holds with $s = 1$. Any attempt to understand the nature and extent of this generalisation should include a description of the possible local structures of the *scale singularity set*, that is the set Σ where s is zero (and where $g^o = s^{-2}g$ is undefined). The main results in this article are some answers to this question and the development of a conformal theory to relate, quite directly, the intrinsic geometric structure of the singularity space Σ to the ambient structure. The classification results for the scale singularity set are new, although simple and elementary. On the other hand the approach to relating the (conformal) geometry of the conformal infinity to the geometry of the ambient structure is more subtle and leads to a number of new results. If s solves (1.1) then so does $-s$, and where s is non-vanishing these solutions determine the same Einstein metric. We shall say that a manifold (M, g) is *almost Einstein* if it admits a covering such that on each open set U of the cover we have that (U, g, s_U) is directed almost Einstein and on overlaps $U \cap V$ we have either $s_U = s_V$ or $s_U = -s_V$. Although there exist almost Einstein spaces which are not directed [3], to simplify the exposition we shall assume here that almost Einstein (AE) manifolds are directed. (So usually we omit the term "directed".) In any case the results apply locally on almost Einstein manifolds which are not directed.

On an Einstein manifold (M, g) the Bianchi identity implies that the scalar curvature Sc^g (i.e. the metric trace of Ric) is constant. Thus simply requiring a metric to be scalar constant is another weakening of the Einstein condition. On compact, connected oriented smooth Riemannian manifolds this may be achieved conformally: this is the outcome of the solution to the "Yamabe problem" due to Yamabe, Trudinger, Aubin and Schoen [4–7]. Just as almost Einstein generalises the Einstein condition, there is a corresponding weakening of the constant scalar curvature condition as follows. We will say that (M, g, s) is a *directed almost scalar constant* structure if $s \in C^\infty(M)$ is a non-trivial solution to the equation $S(g, s) = $ *constant* where

$$S(g, s) = \frac{2}{d} s(J^g - \Delta^g)s - |ds|_g^2. \tag{1.2}$$

Away from the zero set (which again we denote by Σ) of s we have $S(g, s) = \mathrm{Sc}^{g^o}/d(d-1)$ where $g^o := s^{-2}g$. In particular, off Σ, $S(g, s)$ is constant if and only if Sc^{g^o} is constant. The normalisation is so that if g^o is the metric of a space form then $S(g, s)$ is exactly the sectional curvature. We shall say that a manifold (M, g) is *almost Scalar constant* (ASC) if it is equipped with a covering such that on each open set U of the cover we have that (U, g, s_U) is directed ASC, and on overlaps $U \cap V$ we have either $s_U = s_V$ or $s_U = -s_V$. In fact, in line with our assumptions above and unless otherwise mentioned explicitly, we shall assume below that any ASC structure is directed.

As suggested above, closely related to these notions are certain classes of the so-called conformally compact manifolds that have recently been of considerable interest. We recall how these manifolds are usually described. Let M^d be a compact smooth manifold with boundary $\Sigma = \partial M$. A metric g^o on the interior M^+ of M is said to be conformally compact if it extends (with some specified regularity) to Σ by $g = s^2 g^o$ where g is non-degenerate up to the boundary, and s is a non-negative defining function for the boundary (i.e. Σ is the zero set for s, and ds is non-vanishing along Σ). In this situation the metric g^o is complete and the restriction of g to $T\Sigma$ in $TM|_\Sigma$ determines a conformal structure that is independent of the choice of defining function s; then Σ with this conformal structure is termed the conformal infinity of M^+. (This notion had its origins in the work of Newman and Penrose; see the introduction of [8] for a brief review.) If the defining function is chosen so that $|ds|_g^2 = 1$ along M then the sectional curvatures tend to -1 at infinity and the structure is said to be asymptotically hyperbolic (AH) (see [9] where there is a detailed treatment of the Hodge cohomology of these structures and related spectral theory). The model is the Poincaré hyperbolic ball and thus the corresponding metrics are sometimes called Poincaré metrics. Generalising the hyperbolic ball in another way, one may suppose that the interior conformally compact metric g^o is Einstein with the normalisation $\mathrm{Ric}(g^o) = -ng^o$, where $n = d - 1$, and in this case the structure is said to be Poincaré–Einstein (PE); in fact PE manifolds are necessarily asymptotically hyperbolic. Such structures have been studied intensively recently in relation to the proposed AdS/CFT correspondence of Maldacena [10,11], related fundamental geometric questions [12–20], and through connections to the ambient metric of Fefferman–Graham [21,22].

For simplicity of exposition we shall restrict our attention to smooth AE and ASC structures (M^d, g, s); that is (M, g) is a smooth Riemannian manifold of dimension $d \geq 3$ and $s \in C^\infty(M)$ satisfies either (1.1) (the AE case) or (1.2) (for ASC). Let us write $M^\pm$ for the open subset of M on which s is positive or, respectively, negative and, as above, Σ for the scale singularity set. The first main results (proved in Section 2) are the following classifications for the possible submanifold structures of Σ.

Theorem 1.1. *Let (M^d, g, s) be a directed almost scalar constant structure with M connected. If $S(g, s) > 0$ then s is nowhere vanishing and (M, g^o) has constant scalar curvature $d(d-1)S(g, s)$. If $S(g, s) < 0$ then s is non-vanishing on an open dense set and Σ is either empty or else is a smooth hypersurface; On $M \setminus \Sigma$, Sc^{g^o} is constant and equals $d(d-1)S(g, s)$. Suppose M is closed (i.e. compact without boundary) with $S(g, s) < 0$ and $\Sigma \neq \emptyset$. A constant rescaling of s normalises $S(g, s)$ to -1, and then $(M \setminus M^-)$ is a finite union of connected AH manifolds. Similar for $(M \setminus M^+)$.*