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# How to quantize forces (?): An academic essay on how the strings could enter classical mechanics

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Dedicated to Karla and Mario Ziman's on the occasion of their wedding, and to one sunny smiling friend.

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ABSTRACT

Geometrical formulation of classical mechanics with forces that are not necessarily potential-generated is presented. It is shown that a natural geometrical "playground" for a mechanical system of point particles lacking Lagrangian and/or Hamiltonian description is an odd-dimensional line element contact bundle. Time evolution is governed by certain canonical two-form $\Omega$ (an analog of $\mathrm{d}p \wedge \mathrm{d}q - \mathrm{d}H \wedge \mathrm{d}t$), which is constructed purely from forces and the metric tensor entering the kinetic energy of the system. Attempt to "dissipative quantization" in terms of the two-form $\Omega$ is proposed. The Feynman path integral over histories of the system is rearranged to a "world-sheet" functional integral. The "umbilical string" surfaces entering the theory connect the classical trajectory of the system and the given Feynman history. In the special case of potential-generated forces, "world-sheet" approach precisely reduces to the standard quantum mechanics. However, a transition probability amplitude expressed in terms of "string functional integral" is applicable (at least academically) when a general dissipative environment is discussed.



## 1. Introduction

Classical mechanics is the best elaborated and understood part of physics. At the first sight relatively innocent Newton's equation of motion becomes mathematically very interesting and fruitful when one leaves a flat vector space, for example, by imposing a simple set of constraints. Nowadays geometrical description of mechanics is concentrated around beautiful and powerful mathematical artillery, which includes [1–10] symplectic and/or Poisson geometry, contact structures, jet prolongations, Riemannian geometry, variational calculus, ergodic theory and so on. Advantages and disadvantages of any of these approaches are dependent on intended applications and personal preferences and disposals. Some of them are useful, when one goes from classical to quantal. Another, when one wants to pass from the non-relativistic domain to relativistic one. And the third and fourth, when we attempt to generalize discrete system dynamics to the continuum one and/or when the number of particles is so large that some statistical methods should be imposed.

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The aim of the paper is to provide a geometrical picture of classical mechanics for physical systems for which Lagrangian and/or Hamiltonian description is missing. This means that the forces acting within the system are not potential-generated. After explaining the geometry beyond the classical dissipative dynamics we make an attempt at quantization. We avoid to couple the system to an environment and to form one conservative super-system switching on an interaction. Our approach is based solely on the system under study and a dynamics in which dissipative strength effects of the environment are described by a velocity dependent external force. In the case of potential-generated forces, the proposed description becomes equivalent to the standard canonical formalism.

The paper is organized as follows. The first two preliminary subparagraphs deal with some basic facts about the contact geometry and its application in point particle mechanics. We concentrate ourselves on the definition of the line element contact bundle and its internal geometrical structure (smooth atlas, bundleness, canonical distribution, natural lift of curves). The subsequent introduction to mechanics is standard. We just want to remind the reader of basic notation and convince him that to use an extended configuration space and related line element contact structure in mechanics is highly advantageous. The main attention is paid to a correct geometrical setting of forces, i.e. we are looking for the answer to the question: "what type of tensorial quantities are forces in general?" The guiding object of (dissipative) dynamics which governs the time evolution is certain two-form $\Omega$. It is constructed only from forces and kinetic energy. The last paragraph is a rather academic and speculative elaboration on possible quantization in terms of the Feynman functional integral. "Umbilical string" surfaces naturally enter the quantization and transition amplitudes are obtained by a "world-sheet" functional integration.

To be honest and collegial, it is necessary to provide here some standard references on the quantization of dissipative systems. There exist several different approaches [11–19] (explicitly time dependent Hamiltonian, method of dual coordinates, nonlinear Schrödinger equation, method of the loss-reservoir) and an interested reader could try to focus on one of the following keywords: damped harmonic oscillator, Kostin's nonlinear Schrödinger–Langevin equation, Caldirola–Kanai hamiltonian, stochastic quantization, Caldeira–Leggett model.

## 2. Preliminaries: Line element contact bundle and classical mechanics

By a mechanical system we understand throughout the paper a system of particles whose positions and velocities are restricted by a set of *holonomic* and/or *integrable differential* constraints. The constraints as well as exterior forces (which are not supposed to be potential-generated only) can be explicitly time dependent.

The aim of the following subsections is to remind the reader of the geometrical setting that is necessary for the proper description of the time evolution. Hopefully, we will recover soon that a natural "playground" for classical mechanics is the *line element contact bundle* of an extended configuration space.

### 2.1. Line element contact bundle

A beautiful introduction to contact structures in physics with a variety of applications can be found in the William Burke's book [4]. I am very strongly recommending to go through it in detail. Its eloquent motto: *...how in hell you can vary $\dot{q}$ without changing q...* applies also to the following text.

Let $\mathcal{M}$ be an ordinary $(n+1)$-dimensional smooth real manifold and $\gamma_1,\ \gamma_2 : \mathbb{R} \to \mathcal{M}$ two parameterized curves thereon. One says that $\gamma_1$ and $\gamma_2$ are in *contact* at a common point $p \in \gamma_1 \cap \gamma_2 \subset \mathcal{M}$, if their tangent vectors (instant velocities) at that point are proportional to each other. The contactness is obviously a weaker notion than tangentiality.

A *line contact element* at point $p$ is the equivalence class of curves being in contact at $p$. Practically, to give a line contact element means to choose a point $p$ of $\mathcal{M}$ and to fix a one-dimensional subspace (undirected line) $\ell \subset T_p\,\mathcal{M}$. The set of all undirected lines passing through the origin of the tangent space under consideration is the projective space $\mathbb{P}\,(T_p\,\mathcal{M})$. Thus forming a sum:

$$\mathcal{C}\mathcal{M} := \bigcup_{p\in\mathcal{M}} \mathbb{P}\,(T_p\,\mathcal{M}) \equiv (\mathbb{P}\,T)\,\mathcal{M}$$

we get a set of all line contact elements of the manifold $\mathcal{M}$. $\mathcal{C}\mathcal{M}$ is not a structureless object, it inherits smooth structure from $\mathcal{M}$ that turns it into a manifold. Concisely, let $\{\mathcal{O}_\mathfrak{I},\ \varphi_\mathfrak{I}\}_\mathfrak{I}$ be any smooth atlas of $\mathcal{M}$ and $\{T(\mathcal{O}_\mathfrak{I}),\ \Phi_\mathfrak{I}\}_\mathfrak{I}$ induced local trivialization of its tangent bundle $T\,\mathcal{M}$, i.e.

$$\Phi_\mathfrak{I} : T(\mathcal{O}_\mathfrak{I}) \longrightarrow \varphi_\mathfrak{I}(\mathcal{O}_\mathfrak{I}) \times \mathbb{R}^{n+1}, \qquad \big\{p \in \mathcal{O}_\mathfrak{I},\ v \in T_p\,(\mathcal{O}_\mathfrak{I})\big\} \longmapsto \big(q^0, \ldots, q^n | \dot{q}^{\,0} = v^0, \ldots, \dot{q}^{\,n} = v^n\big).$$

Let us, moreover, define the system of:

– open subsets $\mathcal{C}_a(\mathcal{O}_\mathfrak{I}) \subset T(\mathcal{O}_\mathfrak{I})$ ($a$ runs from 0 to $n$):

$$\mathcal{C}_a(\mathcal{O}_\mathfrak{I}) := \Phi_\mathfrak{I}^{-1}\Big\{\text{those points of } \varphi_\mathfrak{I}(\mathcal{O}_\mathfrak{I}) \times \mathbb{R}^{n+1} \text{ whose } a\text{th dot coordinate } \dot{q}^{\,a} \neq 0\Big\}$$

– morphisms $\Phi_{a,\,\mathfrak{I}} :=$ the restriction $\Phi_\mathfrak{I}|_{\mathcal{C}_a(\mathcal{O}_\mathfrak{I})}$.

Then the collection $\{\mathcal{C}_a(\mathcal{O}_\mathfrak{I}),\ \Phi_{a,\,\mathfrak{I}}\}_{a,\,\mathfrak{I}}$ provides a smooth atlas of $\mathcal{C}\mathcal{M}$.