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# Quantum complex Minkowski space

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**Abstract**

The complex Minkowski phase space has the physical interpretation of the phase space of the scalar massive conformal particle. The aim of the paper is the construction and investigation of the quantum complex Minkowski space.


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## 1. Introduction

Extending the Poincaré group by dilation and acceleration transformations, one obtains the conformal group $\mathrm{SU}(2,2)/\mathbb{Z}_4$, which is the symmetry group of the conformal structure of compactified Minkowski space-time $M$, where $\mathbb{Z}_4 = \{i^k \mathrm{id} : k = 0, 1, 2, 3\}$ is the centralizer of $\mathrm{SU}(2, 2)$. According to the prevailing point of view $\mathrm{SU}(2,2)/\mathbb{Z}_4$ is the symmetry group for physical models which describe massless fields or particles, but has no application to the theory of massive objects. However, using the twistor description [1] of Minkowski space-time and the orbit method [2], the different orbits of $\mathrm{SU}(2,2)/\mathbb{Z}_4$ in the conformally compactified complex Minkowski space $\mathbb{M} := M^{\mathbb{C}}$ may be considered to be the classical phase spaces of massless and massive scalar conformal particles, antiparticles and tachyons, see [3,4].

The motivation for various attempts to construct models of non-commutative Minkowski space-time is the belief that this is the proper way to avoid divergences in quantum field theory [5]. Here, on the other hand, our aim is to quantize the classical phase space $\mathbb{M}^{++} \subset \mathbb{M}$ of the massive particle by replacing it by the Toeplitz-like operator $C^*$-algebra $\mathcal{M}^{++}$. To this end we first quantize the

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classical states of the massive scalar conformal particle by constructing the coherent state map $\mathcal{K} : \mathbb{M}^{++} \to \mathbb{CP}(\mathcal{H})$ of $\mathbb{M}^{++}$ into the complex projective Hilbert space $\mathbb{CP}(\mathcal{H})$, i.e. the space of the pure states of the system. In the next step we define the Banach algebra $\overline{\mathcal{P}}^{++}$ of annihilation operators as the ones having the coherent states $\mathcal{K}(m)$, $m \in \mathbb{M}^{++}$, as eigenvectors. Finally, the quantum phase space $\mathcal{M}^{++}$ will be the $C^*$-algebra generated by $\overline{\mathcal{P}}^{++}$.

Let us remark that application of the above method of quantization to the case of $\mathbb{R}^{2N}$ phase space leads to the Heisenberg–Weyl algebra. In our construction the conformal group and $\mathcal{M}^{++}$ are related in exactly the same way as are the Heisenberg group and the Heisenberg–Weyl algebra.

The conformally compactified Minkowski space $M$ can be reconstructed from $\mathcal{M}^{++}$ as the Šilov boundary of the interior of the spectrum of the commutative Banach algebra $\mathcal{P}^{++}$. It can also be considered in the framework of Kostant–Souriau quantization as the $\mathrm{SU}(2,2)/\mathbb{Z}_4$-invariant configuration space for the phase space $T^*M$. Similarly, if we consider the holomorphic model [4], see Sections 2 and 3, the classical conformal phase space $\mathbb{M}^{++}$ has the interpretation of the configuration space constructed by the $\mathrm{SU}(2,2)/\mathbb{Z}_4$-invariant Kähler polarization. In [4] a model of the classical field theory on $\mathbb{M}^{++}$ was proposed. This paper is an effort, developing the results presented in [6], to construct a quantum description of the conformal massive particle, see Section 4. In Section 5 the physical interpretation of the quantum phase space $\mathcal{M}^{++}$ is discussed.

## 2. Complex Minkowski space as the phase space of the conformal scalar massive particle

Following [3,7,4], we present the twistor description of phase spaces of the conformal scalar massive particles. Let us recall that twistor space $\mathbf{T}$ is $\mathbb{C}^4$ equipped with a Hermitian form $\eta$ of signature $(++--)$. The symmetry group of $\mathbf{T}$ is the group $\mathrm{SU}(2,2)$, where $g \in \mathrm{SU}(2,2)$ iff $g^\dagger \eta g = \eta$ and $\det g = 1$.

In relativistic mechanics the elementary phase spaces are given by the coadjoint orbits of the Poincaré group, see [8], which are parametrized in this case by mass, spin, and signature of the energy of the relativistic particle. Similarly, elementary phase spaces for conformal group one identifies with its coadjoint orbits. Since conformal Lie algebra $\mathrm{su}(2,2)$ is simple we will identify its dual $\mathrm{su}(2,2)^*$ with $\mathrm{su}(2,2)$ using Cartan–Killing form:

$$\langle X, Y\rangle = \tfrac{1}{2}\mathrm{Tr}(XY), \tag{2.1}$$

where $X, Y \in \mathrm{su}(2,2)$. Thus the coadjoint representation $\mathrm{Ad}^*_g : \mathrm{su}(2,2)^* \to \mathrm{su}(2,2)^*$ is identified with the adjoint one

$$\mathrm{Ad}_g X = gXg^{-1}, \tag{2.2}$$

where $g \in \mathrm{SU}(2,2)$. For the complete description and physical interpretation of $\mathrm{Ad}^*(\mathrm{SU}(2,2))$-orbits see [9,7].

One defines the compactified complex Minkowski space $\mathbb{M}$ as the Grassmannian of two-dimensional complex vector subspaces $w \subset \mathbf{T}$ of the twistor space and $\mathrm{SU}(2,2)$ acts on $\mathbb{M}$ by

$$\sigma_g : w \to gw. \tag{2.3}$$

The Grassmannian $\mathbb{M}$ splits into the orbits $\mathbb{M}^{kl}$ indexed by the signatures of the restricted Hermitian forms $\operatorname{sign} \eta|_z = (k, l)$, where $k, l = +, -, 0$.