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A new proof of the local regularity of the eta invariant of a Dirac operator

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Abstract

In this paper we use the approach of our earlier proof of the local index theorem to give a new proof of Bismut–Freed's result on the local regularity of the eta invariant of a Dirac operator in odd dimension.

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1. Introduction

The eta invariant of a selfadjoint elliptic ΨDO was introduced by Atiyah–Patodi–Singer [4] as a boundary correction to their index formula on manifolds with boundary. It is obtained as the regular value at $s = 0$ of the eta function,

$$\eta(P; s) = \operatorname{Tr} P|P|^{-(s+1)} = \int_M \eta(P; s)(x). \tag{1.1}$$

However, the residue at $s = 0$ of the local eta function $\eta(P; s)(x)$ needs not vanish (see, e.g., [14]), so it is a nontrivial fact that the regular value exists (see [5,15]). Nevertheless, global K-theoretic arguments allows us to reduce the proof to the case of a Dirac operator on an odd dimensional spin Riemannian manifold with coefficients in a Hermitian vector bundle, for which the result can be obtained by using invariant theory (see [5,16]).

Subsequently, Wodzicki [25,26] generalized the result of Atiyah–Patodi–Singer and Gilkey in the nonselfadjoint setting. More precisely, he proved that:

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(i) The regular value at $s = 0$ of the zeta function $\zeta_\theta(P; s) = \operatorname{Tr} P_\theta^{-s}$ of an elliptic ΨDO is independent of the spectral cutting $\{\arg\lambda = \theta\}$ used to define P_θ^s (see [26, 1.24]);
(ii) The noncommutative residue of a ΨDO projection is always zero (see [26, 7.12]).

The original proofs of Wodzicki are quite involved, but it follows from an observation of Brüning-Lesch that Wodzicki's results can be deduced from the aforementioned result of Atiyah–Patodi–Singer and Gilkey (see [11, Lem. 2.6] and [22, Rmk. 4.5]).

Next, in the case of a Dirac operator $\not{D}_{\mathcal{E}}$ on a odd spin Riemannian manifold M^n with coefficients in a Hermitian bundle $\mathcal{E}$, Bismut–Freed [10] proved in a purely analytic fashion that the local eta function $\eta(P; s)(x)$ is actually regular at $s = 0$. More precisely, by the Mellin formula for $\Re s > -1$ we have

$$\not{D}_{\mathcal{E}}|\not{D}_{\mathcal{E}}|^{-(s+1)} = \Gamma\left(\frac{s+1}{2}\right)^{-1} \int_0^\infty t^{\frac{s-1}{2}} \not{D}_{\mathcal{E}} e^{-t\not{D}_{\mathcal{E}}^2}\, dt. \tag{1.2}$$

For $t > 0$ let $h_t(x, y) \in C^\infty(M, |\Lambda|(M) \otimes \operatorname{End}\mathcal{E})$ be the kernel of $\not{D}_{\mathcal{E}} e^{-t\not{D}_{\mathcal{E}}^2}$, where $|\Lambda|(M)$ denotes the bundle of densities on M. Since by standard heat kernel asymptotics we have $\operatorname{tr}_{\mathcal{E}} h_t(x, x) = O(t^{-n/2})$ as $t \to 0^+$ (see Theorem 2.10 ahead), for $\Re s > n - 1$ we get

$$\eta(\not{D}_{\mathcal{E}}; s)(x) = \Gamma\left(\frac{s+1}{2}\right)^{-1} \int_0^\infty t^{\frac{s-1}{2}} \operatorname{tr}_{\mathcal{E}} h_t(x, x)\, dt. \tag{1.3}$$

Then Bismut–Freed ([10, Thm. 2.4]) proved that in the C^0-topology we have

$$\operatorname{tr}_{\mathcal{E}} h_t(x, x) = O(\sqrt{t}) \qquad \text{as } t \to 0^+. \tag{1.4}$$

It thus follows that the local eta function $\eta(\not{D}_{\mathcal{E}}; s)(x)$ is actually holomorphic for $\Re s > -2$. In particular, we have the formula,

$$\eta(\not{D}_{\mathcal{E}}) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \operatorname{Tr} \not{D}_{\mathcal{E}} e^{-t\not{D}_{\mathcal{E}}^2}\, dt, \tag{1.5}$$

which, for instance, plays a crucial role in the study of the adiabatic limit of the eta invariant of a Dirac operator (see [9,10]). Incidentally, Bismut–Freed's asymptotics (1.4), which is a purely analytic statement, implies the global regularity of the eta invariant of any selfadjoint elliptic ΨDO, as well as the aforementioned generalizations of Wodzicki.

Now, the standard proof of the asymptotics (1.4) is essentially based on a reduction to the local index theorem of Patodi, Gilkey, Atiyah–Patodi–Singer ([1,16]; see also [13]), which provides us with a heat kernel proof of the index theorem of Atiyah–Singer [2,3] for Dirac operators. In the original proof of (1.4) in [10] the reduction is done by introducing an extra Grassmanian variable z, $z^2 = 0$, and in [18, Sect. 8.3] by means of a suspension argument. Moreover, in [10] the authors refer to the results of [17] to justify the differentiability of the heat kernel asymptotics.

On the other hand, in [20] the approach to the heat kernel asymptotics of [17] was combined with general considerations on Getzler's order of Volterra ΨDO's to produce a new short proof of the local index theorem which holds in many other settings. Furthermore, the arguments used in this proof have other applications such as the computation of the CM cyclic cocycle of [12] for Dirac operators.

The aim of this paper is to show that we can get a direct proof of Bismut–Freed's asymptotics by using the approach of [20]. In fact, once the background from [17,20] is set-up, the proof becomes extremely simple since it is deduced by combining the considerations on Getzler orders