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Killing forms on G_2- and Spin_7-manifolds

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Abstract

Killing forms on Riemannian manifolds are differential forms whose covariant derivative is totally skew-symmetric. We prove that on a compact manifold with holonomy G_2 or Spin_7 any Killing form has to be parallel. The main tool is a universal Weitzenböck formula. We show, how such a formula can be obtained for any given holonomy group and any representation defining a vector bundle.

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1. Introduction

Killing forms are a natural generalization of Killing vector fields. They are defined as differential forms u, such that ∇u is totally skew-symmetric. More generally one considers twistor forms, as forms in the kernel of an elliptic differential operator, defined similarly to the twistor operator in spin geometry. Twistor 1-forms are dual to conformal vector fields. Killing forms are coclosed twistor forms.

The notion of Killing forms was introduced by Yano in [17], where he already noted that a p-form u is a Killing form if and only if for any geodesic γ the $(p-1)$-form $\dot{\gamma} \lrcorner u$ is parallel along γ. In particular, Killing forms define quadratic first integrals of the geodesic equation, i.e. functions, which are constant along geodesics. This motivated an intense study of Killing forms in the physics literature, e.g. in the article of Penrose and Walker [12]. More recently, Killing and twistor forms have been successfully applied to define symmetries of field equations (cf. [4,5]).

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On the standard sphere, the space of twistor forms coincides with the eigenspace of the Laplace operator for the minimal eigenvalue and Killing forms are the coclosed minimal eigenforms. The sphere also realizes, the maximal possible number of twistor or Killing forms. So far, there are only very few further examples of compact manifolds admitting Killing p-forms with $p \geq 2$. These are Sasakian, nearly Kähler and weak-G_2 manifolds, and products of them. The Killing p-forms of these examples satisfy an additional equation and it turns out that they are in 1-1 correspondence to parallel $(p+1)$-forms on the metric cone. In particular, they only exist on manifolds with Killing spinors (cf. [2,13]).

The present article is the last step in the study of Killing forms on manifolds with restricted holonomy. It was already known that on compact Kähler manifolds Killing p-forms with $p \geq 2$ are parallel [16]. Moreover, we showed in Ref. [11,3] that the same is true on compact quaternion-Kähler manifolds and compact symmetric spaces. Here, we will prove the corresponding statement for the remaining holonomies G_2 and Spin_7.

The Hodge $*$-operator preserves the space of twistor forms. In particular, it maps Killing forms to closed twistor forms, which we will call $*$-Killing forms.

Theorem 1.1. *Let (M^7, g) be a compact manifold with holonomy G_2. Then, any Killing form and any $*$-Killing form is parallel. Moreover, any twistor p-form, with $p \neq 3, 4$, is parallel.*

Theorem 1.2. *Let (M^8, g) be a compact manifold with holonomy* Spin_7. *Then, any Killing form and any $*$-Killing form is parallel. Moreover, any twistor p-form, with $p \neq 3$–5, is parallel.*

The main tool for proving the two theorems are suitable Weitzenböck formulas for the irreducible components of the form bundle. More generally, we prove a universal Weitzenböck formula, i.e. we show, how to obtain for any fixed holonomy group G and any irreducible G-representation π, a Weitzenböck formula for certain first order differential operators acting on sections of the vector bundle defined by π. Our formula is already known in the case of Riemannian holonomy SO_n (cf. [8]). However, it seems to be new and so far unused in the case of the exceptional holonomies G_2 and Spin_7. We describe, here, an approach to Weitzenböck formulas which, is further developed and completed in Ref. [15].

2. Twistor forms on Riemannian manifolds

In this section, we recall the definition and basic facts on twistor and Killing forms. More details and further references can be found in Ref. [13]. Most important for the later application will be the integrability condition given in Proposition 2.2.

Consider a n-dimensional Euclidean vector space $(V, \langle\cdot,\cdot\rangle)$. Then, the tensor product $V^* \otimes \Lambda^p V^*$ has the following $O(n)$-invariant decomposition:

$$V^* \otimes \Lambda^p V^* \cong \Lambda^{p-1} V^* \oplus \Lambda^{p+1} V^* \oplus \Lambda^{p,1} V^*,$$

where $\Lambda^{p,1} V^*$ is the intersection of the kernels of wedge and inner product. This decomposition immediately translates to Riemannian manifolds (M^n, g), where we have:

$$T^*M \otimes \Lambda^p T^*M \cong \Lambda^{p-1} T^*M \oplus \Lambda^{p+1} T^*M \oplus \Lambda^{p,1} T^*M, \tag{1}$$

with $\Lambda^{p,1} T^*M$ denoting the vector bundle corresponding to the representation $\Lambda^{p,1}$. The covariant derivative $\nabla\psi$ of a p-form ψ is a section of $T^*M \otimes \Lambda^p T^*M$. Its projections onto the summands $\Lambda^{p+1} T^*M$ and $\Lambda^{p-1} T^*M$ are just the differential $\mathrm{d}\psi$ and the codifferential $\mathrm{d}^*\psi$. Its projection onto the third summand $\Lambda^{p,1} T^*M$ defines a natural first order differential operator T, called the