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Central extensions of gauge transformation groups of higher abelian gerbes

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Abstract

We construct a central extension of the smooth Deligne cohomology group of a compact oriented odd dimensional smooth manifold, generalizing that of the loop group of the circle. While the central extension turns out to be trivial for a manifold of dimension $3, 7, 11, \ldots$, it is non-trivial for $1, 5, 9, \ldots$. In the case where the central extension is non-trivial, we show an analogue of the Segal–Witten reciprocity law.

Keywords: Smooth Deligne cohomology; Central extensions of groups; Gerbes

1. Introduction

The *smooth Deligne cohomology* [4,9,11] of a smooth manifold X is defined to be the hypercohomology $H^q(X, \mathbb{Z}(p)^\infty_D)$ of the complex of sheaves

$$\mathbb{Z}(p)^\infty_D : \mathbb{Z} \to \underline{A}^0 \xrightarrow{d} \underline{A}^1 \xrightarrow{d} \cdots \xrightarrow{d} \underline{A}^{p-1} \to 0 \to \cdots,$$

where $\mathbb{Z}$ is the constant sheaf, located at degree 0, and $\underline{A}^q$ is the sheaf of germs of $\mathbb{R}$-valued differential q-forms. If X is compact, then we can identify the abelian group $H^p(X, \mathbb{Z}(p)^\infty_D)$ with $(A^{p-1}(X)/A^{p-1}(X)_\mathbb{Z}) \times H^p(X, \mathbb{Z})$ by a non-canonical isomorphism, where $A^{p-1}(X)$ is the group of $(p-1)$-forms on X, and $A^{p-1}(X)_\mathbb{Z}$ is the subgroup consisting of closed integral $(p-1)$-forms.

Heuristically, we can interpret $H^{n+2}(X, \mathbb{Z}(n+2)^\infty_D)$ as the group classifying "abelian n-gerbes with connection on X". Besides the interpretation, we can also interpret $H^{n+1}(X, \mathbb{Z}(n+1)^\infty_D)$ as the "gauge transformation group of an abelian n-gerbe on X". For example, if $n = 0$, then

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$H^1(X, \mathbb{Z}(1)^\infty_D)$ is naturally isomorphic to the group $C^\infty(X, \mathbb{T})$ of smooth functions on X with values in the unit circle $\mathbb{T} = \{u \in \mathbb{C} | |u| = 1\}$. The group is nothing but the gauge transformation group of a principal $\mathbb{T}$-bundle (0-gerbe) over X.

Gauge transformation groups of principal bundles are fundamental ingredients not only in mathematics but also in physics. Hence it would be meaningful to study smooth Deligne cohomology groups as gauge transformation groups of higher abelian gerbes.

On the circle S^1, the gauge transformation group of a principal bundle with connected structure group is often identified with a loop group. As is well known [22], the loop group of a compact Lie group has non-trivial central extensions by $\mathbb{T}$. Let $\widehat{L\mathbb{T}}$ denote the *universal central extension* of the loop group $L\mathbb{T} = C^\infty(S^1, \mathbb{T})$. In [6], Brylinski and McLaughlin provided geometric constructions of a central extension of $L\mathbb{T} = H^1(S^1, \mathbb{Z}(1)^\infty_D)$ which is isomorphic to $\widehat{L\mathbb{T}}/\mathbb{Z}_2$. We can describe one of their constructions, using only basic tools of smooth Deligne cohomology. As a straight generalization of the description, we can construct a central extension of $H^{n+1}(M, \mathbb{Z}(n+1)^\infty_D)$ by $\mathbb{T}$ for any compact oriented smooth $(2n+1)$-dimensional manifold M, which is the subject of this paper.

We explain the construction more precisely. The basic tools we use are the *cup product* and the *integration* for smooth Deligne cohomology. Let X be a smooth manifold. The cup product is a natural homomorphism

$$\cup : H^p(X, \mathbb{Z}(p)^\infty_D) \otimes_{\mathbb{Z}} H^q(X, \mathbb{Z}(q)^\infty_D) \to H^{p+q}(X, \mathbb{Z}(p+q)^\infty_D)$$

which is associative and (graded) commutative [4,11]. For the integration to be considered, we suppose that X is a compact oriented smooth d-dimensional manifold without boundary. Then the integration is a natural homomorphism:

$$\int_X : H^{d+1}(X, \mathbb{Z}(d+1)^\infty_D) \to \mathbb{R}/\mathbb{Z}.$$

By using Čech cohomology, we can describe the operations above explicitly.

Let n be a non-negative integer fixed. We put $\mathcal{G}(X) = H^{n+1}(X, \mathbb{Z}(n+1)^\infty_D)$ for a smooth manifold X. For a compact oriented smooth $(2n+1)$-dimensional manifold M without boundary, a group 2-cocycle $S_M : \mathcal{G}(M) \times \mathcal{G}(M) \to \mathbb{R}/\mathbb{Z}$ is given by $S_M(f, g) = \int_M f \cup g$. Now we define the central extension $\tilde{\mathcal{G}}(M)$ to be the set $\mathcal{G}(M) \times \mathbb{T}$ endowed with the group multiplication:

$$(f, u) \cdot (g, v) = (f + g, uv \exp 2\pi \mathrm{i} S_M(f, g)).$$

In general, we can make $\mathcal{G}(M)$ and $\tilde{\mathcal{G}}(M)$ into infinite dimensional Lie groups. However, to avoid discussion not essential to our aim, we will not treat such Lie group structures in this paper.

One result of this paper is the (non-)triviality of $\tilde{\mathcal{G}}(M)$ as a central extension of $\mathcal{G}(M)$.

Theorem 1. *Let n be a non-negative integer, and M a compact oriented smooth $(2n+1)$-dimensional manifold without boundary.*

(a) *If $n = 2k$, then $\tilde{\mathcal{G}}(M)$ is non-trivial.*
(b) *If $n = 2k+1$, then $\tilde{\mathcal{G}}(M)$ is trivial.*

As a consequence, we obtain non-trivial central extensions in the case that the dimension of M is $1, 5, 9, \ldots$. In particular, when $n = 0$ and $M = S^1$, we recover the central extension $\widehat{L\mathbb{T}}/\mathbb{Z}_2$.