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A conservation law model for bidensity suspensions on an incline

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HIGHLIGHTS

- A model for bidensity suspensions of a viscous fluid on an incline is proposed.
- Up to moderate concentrations, particles and fluid separate into distinct fronts.
- We derive the long-term behavior of the front positions and compare to experiments.

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ABSTRACT

We study bidensity suspensions of a viscous fluid on an incline. The particles migrate within the fluid due to a combination of gravity-induced settling and shear induced migration. We propose an extension of a recent model (Murisic et al., 2013) for monodisperse suspensions to two species of particles, resulting in a hyperbolic system of three conservation laws for the height and particle concentrations. We analyze the Riemann problem and show that the system exhibits three-shock solutions representing distinct fronts of particles and liquid traveling at different speeds as well as singular shock solutions for sufficiently large concentrations, for which the mechanism is essentially the same as the single-species case. We also consider initial conditions describing a fixed volume of fluid, where solutions are rarefaction–shock pairs, and present a comparison to recent experimental results. The long-time behavior of solutions is identified for settled mono- and bidisperse suspensions and some leading-order asymptotics are derived in the single-species case for moderate concentrations.

1. Introduction

Non-colloidal suspensions of particles in a shear flow exhibit complex interactions within the mixture. These suspensions have many applications, for example in modeling spiral separators used in the mining industry [1]. For suspensions on an incline, the competing effects of settling due to gravity and shear-induced migration lead to an interesting phenomenon [2,3]: for low concentrations, there is a 'settled' regime in which the particles and fluid separate into distinct fronts, while for sufficiently high concentrations there is a 'ridged' regime in which only a single particle-rich front appears. Murisic et al. [4,5] developed a model for monodisperse suspensions on an incline based on the diffusive flux model of Acrivos [6,7], which was successfully used to predict the experimentally observed settled and ridged regimes and the time evolution of the fronts. In this model the particles are assumed to be in equilibrium in the normal direction and the leading order system for the film height and particle concentrations form a hyperbolic system of conservation laws with fluxes determined by a system of ODEs that describe the distribution of particles in the normal direction. Analysis of hyperbolic systems arising for bidisperse settling has been extensively studied in the context of other models, for instance in the case of settling in a quiescent fluid [8], though not in the incline geometry where we are interested in the long-term dynamics of the fronts rather than the settling of particles to the substrate. The Riemann problem has been studied for the monodisperse model [9,10], showing the existence of double shock solutions in the settled regime and a transition to singular shocks that occurs in the ridged regime, where particles accumulate at the front. The corresponding rarefaction–singular shock solutions that arise for constant volume initial conditions have also been studied [11].

In this work, we propose an extension of the model to bidisperse suspensions by employing a modification of the diffusive flux model to multiple species [12,13] and present some preliminary with comparison to recent experiments that build on previous work to identify the qualitative behavior of the fronts in the incline problem [14]. In Section 2, we present the model and

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discuss the details of the particle fluxes. In Section 3, we describe the Riemann problem for the system. The essential difference from the single species problem is the presence of an additional trailing shock delimiting the transition from heavier to lighter particles, while the remaining two shocks have a similar structure to their monodisperse counterparts. Due to the increased complexity of the system, it is more difficult to obtain analytical results, so we describe the system qualitatively and through numerical simulations. In Section 4, we consider fixed-volume solutions corresponding to results from experiments in which the particles settle to the substrate. We note that the Riemann problem is still relevant here to describe local shock solutions. The full solution has the form of a sequence of rarefaction–shock pairs corresponding to different fronts for the particles and fluid. We derive theoretical results for the long-time behavior of monodisperse suspensions in the settled regime, where the concentration of particles uniformly approaches a critical concentration independent of initial conditions. The asymptotic behavior of the front positions is derived, extending the existing results for the high-concentration [11,10] and dilute limit [5]. Then, some results are then extended to the bidisperse problem. Finally, in Section 5, we present a qualitative comparison of the front positions to recent experiments for the bidisperse problem.

2. Model

Our goal is to derive a system of hyperbolic conservation laws governing the evolution of the film height and particle concentrations for a bidensity suspension. This follows the standard lubrication approach for thin viscous films, but is complicated by the dependence of the velocity profiles on the particle distributions. Our model is an extension of the dynamic model proposed by Murisic et al. [5] to multiple species of particles, making use of the equilibrium theory for the diffusive flux model for bidisperse suspensions [13,15]. We consider a mixture of a fluid with large viscosity μ_ℓ and density ρ_ℓ and two species of negatively buoyant particles with diameter d and densities $\rho_{p,1}$, $\rho_{p,2}$ satisfying $\rho_\ell < \rho_{p,1} < \rho_{p,2}$. The suspension is allowed to spread down an incline of angle α. The geometry of the system is summarized in Fig. 1; we restrict our attention to the two-dimensional problem where the fluid does not vary in the span-wise direction of the incline. The mixture is assumed to be Newtonian with an effective viscosity $\mu(\phi)$ that depends on the particle concentration ϕ and density $\rho(\phi) = (1-\phi)\rho_\ell + \phi_1\rho_{p,1} + \phi_2\rho_{p,2}$. The equations for suspension momentum and particle mass conservation are

$$\rho\frac{\partial \boldsymbol{u}}{\partial t} + \rho\boldsymbol{u}\cdot\nabla\boldsymbol{u} = \nabla\cdot\left(-pI + \mu(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)\right) + \rho\mathbf{g}, \tag{1}$$

$$0 = \frac{\partial \phi_i}{\partial t} + \boldsymbol{u}\cdot\nabla\phi_i + \nabla\cdot\boldsymbol{J}_i, \quad i = 1, 2, \tag{2}$$

with $\boldsymbol{u} = (u, w)$. The mixture viscosity is given by the Krieger–Dougherty relation $\mu(\phi) = \mu_\ell(1-\phi/\phi_{\max})^{-2}$.

The derivation of the conservation law model uses the standard thin-film approximation to reduce (1) to an equation for the film height and is summarized here, following [5]. The equations are non-dimensionalized according to the scales

$$(\hat{x}, \hat{z}) = H(x/\epsilon, z), \qquad (\hat{u}, \hat{w}) = \frac{H^2\rho_\ell g\sin\alpha}{\mu_\ell}(u, \epsilon w),$$

$$\hat{p} = \frac{U\mu_\ell}{H}p, \qquad (\hat{J}^{(x)}, \hat{J}^{(z)}) = \frac{d^2 U}{H^2}(\epsilon J^{(x)}, J^{(z)})$$

where $U = H^2\rho_\ell g\sin\alpha/\mu_\ell$ and $\epsilon \ll 1$. Hereafter, all variables are non-dimensionalized and the hats are dropped for brevity. To simplify the model, we now make the assumption that the particles

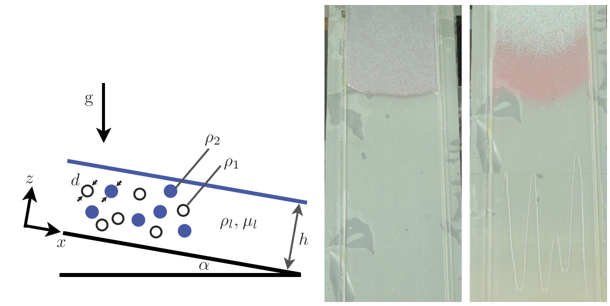

Fig. 1. Left: schematic for the incline problem. Center and right: images from a typical experiment [16] with $\alpha = 20°$, concentrations $\phi_1 = \phi_2 = 0.15$ of each particle type and at times $t = 60$ s and $t = 800$ s. The dimensions in the image are 0.14 m $\times$ 0.5 m.

equilibrate quickly in the normal direction compared to the fluid flow down the incline, which requires that $\epsilon \ll (d/H)^2 \ll 1$. The separation of time scales allows the dynamics in the z and x-directions to be considered separately. In the z-direction, the particles are always in equilibrium, i.e.

$$\boldsymbol{J}_i^{(z)} = 0, \quad i = 1, 2. \tag{3}$$

To leading order in ϵ, the non-dimensionalized x-component of (1) becomes

$$(\mu u_z)_z = -\rho. \tag{4}$$

We impose the no-slip boundary condition on the incline surface and stress-balance on the free surface:

$$u\big|_{z=0} = 0, \qquad \mu u_z\big|_{z=h} = 0. \tag{5}$$

We want to integrate in z to obtain equations for the film height $h(x, t)$ and depth-averaged concentrations

$$\phi_{0,i}(x, t) = \frac{1}{h}\int_0^h \phi_i(x, z', t)\,\mathrm{d}z'. \tag{6}$$

It will also be convenient to define $\phi_0 = \phi_{0,1} + \phi_{0,2}$ and $X_0 = \phi_{0,1}/\phi_0$. Conservation of mass for the fluid and particles, to leading order, is the system

$$0 = \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\int_0^h u\,\mathrm{d}z\right), \tag{7a}$$

$$0 = \frac{\partial (h\phi_{0,i})}{\partial t} + \frac{\partial}{\partial x}\left(\int_0^h u\phi_i\,\mathrm{d}z\right), \quad i = 1, 2. \tag{7b}$$

It remains to find the x-velocity profile $u(x, z, t)$. For a clear fluid, this would be computed explicitly by integrating (4) with the boundary conditions (5), but the dependence on the concentration ϕ (through ρ) means that (3) must also be used to determine the particle distribution in z. Eqs. (3) and (4) form a system of ordinary differential equations in z for the particle concentration ϕ, proportion of lighter particles $X = \phi_1/\phi$ and shear stress $\sigma = \mu(\phi)\frac{\partial u}{\partial z}$. To remove the dependence on the film height $h(x, t)$ we use the scaled variable $s = z/h$ and scaled quantities $\widetilde{\sigma}(x, s, t) = h^{-1}\sigma(x, hs, t)$, $\widetilde{\phi}(x, s, t) = \phi(x, sh, t)$ and x-velocity $\widetilde{u}(x, s, t) = h^{-2}u(x, sh, t)$.

By rewriting the integrals in terms of $s = z/h$ and using the solution to the equilibrium system (10), we can write the system (7) in the more useful form

$$0 = \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(h^3 f(\phi_0, X_0)), \tag{8a}$$

$$0 = \frac{\partial (h\phi_{0,i})}{\partial t} + \frac{\partial}{\partial x}(h^3 g_i(\phi_0, X_0)), \quad i = 1, 2. \tag{8b}$$