Pergamon

*Chaos, Solitons & Fractals* Vol. 4, No. 11, pp. 2015–2041, 1994

0960-0779/94$7.00 + .00

0960–0779(94)E0080–9

# Periodic and Chaotic Solutions for a Nonlinear System Arising from a Nuclear Spin Generator

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(*Received* 25 *February* 1994)

**Abstract**—We study in great detail a system of three first-order ordinary differential equations describing the behaviour of a nuclear spin generator (NSG). This system, not much referred to in literature, displays a large variety of behaviours, both regular and chaotic. Existence of periodic solutions is proved for certain ranges of parameters. Stability criteria for periodic solutions are given. The system is shown here to have a codimension-two bifurcation point: a Takens–Bogdanov bifurcation point. Chaotic behaviours arise from (i) period-doubling bifurcations, (ii) intermittency route, and (iii) homoclinic bifurcations. The gluing of strange attractors and their ungluing, with periodic and chaotic behaviours in the intervening parametric range, not reported earlier for any chaotic system, are shown to occur for the NSG system. Also, in certain parameteric intervals, coexisting attractors and coexisting strange attractors are found to occur. In view of the larger variety of phenomena exhibited by NSG in comparison with the Lorenz system, it is claimed that the former is a better archetypal system for chaos.

## 1. INTRODUCTION

In this paper, we discuss regular and chaotic flow of a third-order system of nonlinear ordinary differential equations which describes the behaviour of a typical nuclear spin generator (NSG). NSG is a high-frequency oscillator which generates and controls the oscillations of the motion of a nuclear magnetization vector in a magnetic field.

If a sample of matter containing proper atomic nuclei is placed in a magnetic field $H_0$, then a net macroscopic nuclear magnetization will exist. When perturbed from equilibrium position by an applied radio-frequency field along a direction perpendicular to the main field, the nuclear magnetization vector will precess about the magnetic field with frequency $|\gamma H_0|$ ($\gamma$ is the gyromagnetic ratio), the Larmor frequency, while decaying to its equilibrium position. The decay is due to damping or relaxing caused by random fluctuations of the local magnetic field of the nuclei in the sample.

The purpose of the NSG is to overcome this damping and maintain the precession of the nuclear magnetization vector about the magnetic field. Now we describe how this may be accomplished in a physical system. The system consists of a suitable sample of matter containing proper nuclei in a relatively strong magnetic field, the equilibrium field $H_0$, defining $z$-direction, an exciting coil with axis in the $x$-direction, perpendicular to $z$, a pick-up coil with axis in the $y$-direction, perpendicular to both $x$ and $z$, and a high-gain amplifier feeding the voltage induced in the pick-up coil back to the exciting coil. The behaviour of the nuclear magnetization vector under the applied magnetic field is given by the Bloch equations; the normalized form of the nuclear spin system is

$$x_1' = -\beta x_1 + x_2, \tag{1}$$

$$x_2' = -x_1 - \beta x_2(1 - kx_3), \tag{2}$$

$$x_3' = \beta(\alpha(1 - x_3) - kx_2^2), \tag{3}$$

where $' \equiv \mathrm{d}/\mathrm{d}t$, $x_1$, $x_2$, $x_3$, are the components of the nuclear magnetization vector in the $x$, $y$, $z$-directions, respectively. $\alpha$, $\beta$, and $k$ are parameters where $\beta\alpha \geqslant 0$, and $\beta \geqslant 0$ are linear damping terms, while the nonlinearity parameter $\beta k$ is proportional to the amplifier gain in the voltage feedback. Physical considerations limit the parameter $\alpha$ to the range $0 < \alpha \leqslant 1$. The mathematical formulation of the problem may be found in Sherman [1].

The physical motivation for the study of the system (1)–(3) is to find the values of $k$, for given $\alpha$ and $\beta$, identified from the sample, for which there is a continued precession or oscillation. The NSG system has application in the investigation of the use of nuclear magnetic resonance in a device for sensing rotation [2].

The system (1)–(3) does not seem to have attracted much attention. Sherman gave a good qualitative analysis for the regular solutions of the NSG system. The purpose of the present paper is to study the system (1)–(3) both analytically and numerically. We find not only the regular solutions analysed by Sherman but a whole gamut of chaotic solutions. It is curious that the original paper with the NSG equations and Lorenz's paper were published in the same year (1963). The NSG system, as we shall show, displays much richer structures than Lorenz system, but, for reasons of history, the latter occupies a more important place. We believe that the NSG system is one of the simplest third-order systems exhibiting a large variety of chaotic behaviours, and may attract more attention in future investigations.

The scheme of the present paper is as follows. Section 2 details some simple properties of the system (1)–(3). Equiliburium points of this system and their linear stability are discussed in Section 3. In Section 4 we prove the existence of periodic solutions about the equiliburium points of the system (1)–(3) using Hopf's theorem. Also, we establish stability conditions using the criterion given by Hsü and Kazarinoff [3] for $x \in R^3$. In Section 5, we give a perturbation solution, and discuss the existence of periodic solutions, using Melnikov subharmonic functions, when $\beta \ll 1$. A detailed numerical study of (1)–(3) is presented in Section 6. Transitions to chaotic attractors via a sequence of period-doubling bifurcations starting with a Hopf bifurcation are found to occur in the system (1)–(3) in the parametric space $\alpha < 1$, $\beta < 1$ and $k > 2$, for some particular choices of the pair $(\alpha, \beta)$. For $k = 2$, with $\beta = 1$ and $\alpha \leqslant 1$, the Hopf and pitchfork bifurcation coincide, with a pair of zero eigenvalues for the equilibrium point $A$. Thus the system (1)–(3) is shown to have a codimension-two bifurcation point: a Takens–Bogdanov bifurcation [4], at which a gluing bifurcation [5] results; as the parameters $k$ and $\beta$ are changed, the gluing bifurcation is replaced by a pair of homoclinic explosions. The analysis of such a point has been presented by Lyubimov and Zaks [6]. We also show that some chaotic solutions are associated with homoclinic connection to a saddle, in the manner of the Lorenz system [7]. Many other chaotic features, analogous to those for the Lorenz system, are discovered for the system (1)–(3); the latter enjoys additional features, such as coexisting strange attractors and gluing of strange attractors, not observed in the study of Lorenz system. Also, the gluing and ungluing of strange attractors which have not been reported for any third-order system are shown to occur for the NSG system.

## 2. SOME QUALITATIVE PROPERTIES

The system (1)–(3) enjoys the symmetry

$$(x_1, x_2, x_3) \to (-x_1, -x_2, x_3).$$

The $x_3$-axis is invariant. All trajectories, which start on the $x_3$-axis remain on it and tend to $(0, 0, 1)$ as $t \to \infty$. Moreover, all trajectories which rotate around the $x_3$-axis do so in a