Physica D 328–329 (2016) 1–8

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Dynamics of curved fronts in systems with power-law memory

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A R T I C L E I N F O

Article history:
Received 10 January 2016
Accepted 5 April 2016
Available online 13 April 2016
Communicated by T. Wanner

Keywords:
Phase transitions
Subdiffusion
Fractional differential equations
Front dynamics
Curvature flow

A B S T R A C T

The dynamics of a curved front in a plane between two stable phases with equal potentials is modeled via two-dimensional fractional in time partial differential equation. A closed equation governing a slow motion of a small-curvature front is derived and applied for two typical examples of the potential function. Approximate axisymmetric and non-axisymmetric solutions are obtained.

1. Introduction

The evolution of phase-transition fronts has been the subject of numerous investigations, due to its technological importance and nontrivial nonlinear dynamics (see, e.g., [1,2]). An efficient tool for description of the front dynamics is the phase-field approach, which is based on the consideration of the evolution of the order parameter field governed by partial differential equations [3–5], which are similar to reaction–diffusion equations describing the propagation of a chemical reaction front [6,7].

If the physical process is characterized by some "hidden" internal variables, their slow relaxation can lead to a temporal nonlocality (memory) in the governing equations (see, e.g., [8–10]). In the simplest case, the evolution of the order parameter $u(\mathbf{x},t)$ is governed by a partial integro-differential equation,

$$\partial_t u = \int_{-\infty}^{t} a(t-\tau)[\Delta u + f(u)](\tau)\mathrm{d}\tau, \tag{1}$$

where zeros of function $f(u)$ correspond to homogeneous phases in the system. If the memory kernel $a(\tau)$ is a non-singular function, which decreases with τ sufficiently fast, the analysis of the front dynamics can be carried out by means of asymptotic methods, and a closed differential equation governing the front shape can be derived [11–13].

Also, Eq. (1) describes reaction–diffusion phenomena in systems with memory. The latter kind of problems are often characterized by a slowly decaying and singular memory kernels which correspond to the phenomenon of *subdiffusion* [14,15]. A typical model of that phenomenon includes a fractional order derivative in time [16,17]:

$$\partial_t^\alpha u = \Delta u + f(u), \quad 0 < \alpha < 1, \tag{2}$$

where

$$\partial_t^\alpha u(\mathbf{x},t) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{t} \frac{\partial_\tau u(\mathbf{x},\tau)}{(t-\tau)^\alpha}\mathrm{d}\tau \tag{3}$$

is the Caputo fractional derivative. Specifically, the subdiffusion is significant in phase transitions when a glass phase is involved [18,19].

In the present paper, we apply model (2) for studying the dynamics of a slowly moving, weakly curved front between two phases with equal thermodynamical potentials. Another interpretation of the same mathematical model is the propagation of the reaction front in a bistable subdiffusion–reaction system. In Section 2, the formulation of problem is given. In Section 3, we investigate the motion of a circular front, and an asymptotic approach is used. In Section 4, we study the dynamics of a front of an arbitrary shape. Section 5 contains some conclusions.

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http://dx.doi.org/10.1016/j.physd.2016.04.004
0167-2789/

2. Formulation of problem

We consider Eq. (2) in an infinite plane written in polar coordinates,

$$\partial_t^\alpha u(r,\theta,t) = \partial_r^2 u + \frac{1}{r}\partial_r u + \frac{1}{r^2}\partial_\theta^2 u + f(u). \tag{4}$$

It is convenient to use the representation,

$$\partial_t^\alpha u(r,\theta,t) = \frac{-1}{\Gamma(-\alpha)}\int_0^\infty [u(r,\theta,t) - u(r,\theta,t-\tau)]\frac{d\tau}{\tau^{\alpha+1}}$$

for the Caputo derivative, which is equivalent to the standard definition (3). We assume that $f(u) = -F'(u)$, where $F(u)$ is the potential which has two minima at $u = u_\pm$ (a maximum of $F(u)$ takes place for an intermediate value of u). These minima of the potential correspond to homogeneous states of the system. At $u = u_\pm, f(u_\pm) = 0$ and $f'(u_\pm) < 0$. Later on, we assume that

$$F(u_+) = F(u_-),$$

see Fig. 1. The typical examples are (i) $f(u) = u - u^3$, which corresponds to a subdiffusive Allen–Cahn equation [20,21]; (ii) $f(u) = \text{sign}(u)(1 - |u|)$, which has been used for finding an exact front solution in the one-dimensional case [20]. In the case of a reaction–subdiffusion problem, $f(u)$ is the reaction function. Later on, we assume that $u_+ > 0$, $u_- < 0$, and define the "front" between two phases, $r = \rho(\theta,t)$, by the relation $u(\rho(\theta,t),\theta,t) = 0$. We consider the case where $u < 0$ as $r < \rho(\theta,t)$, and $u > 0$ as $r > \rho(\theta,t)$, and apply the boundary condition

$$u(r \to \infty) = u_+.$$

We assume that $\rho(\theta,0) = O(\epsilon^{-1}) \gg 1$, and the initial value of $u(r,\theta,t)$ is close to u_- for $r < \rho(\theta,t)$ and to u_+ for $r > \rho(\theta,t)$, except a transition layer of the width $O(1)$ ("domain wall"). Thus, we consider a large "island" of the negative phase in an infinite "ocean" of the positive phase, the opposite configuration can be considered in a similar way.

It is well known that in the case of a one-dimensional front between two phases with equal potentials, which occupy two semi-planes, the solution tends to a stationary one, i.e., the front is motionless [20]. As to a curved front in a plane, we expect that the motion will be slow in the case of a small curvature, similarly to the case of the normal diffusion [6]. That allows to apply asymptotic methods for the simplification of the problem.

3. Axisymmetric solution

First, let us consider an axisymmetric solution, $u = u(r,t)$. In that case the front has a circular shape: $r = \rho(t)$, where $u(\rho(t),t) = 0$. Assume that the radius of the front is large,

$$\rho(t) = \epsilon^{-1}R(t), \quad \text{where } \epsilon \ll 1 \text{ and } R(t) = O(1),$$

and introduce the time scaling,

$$(\tilde{t},\tilde{\tau}) = \epsilon^{1/\alpha}(t,\tau). \tag{5}$$

The width of the transition zone between phases is $O(1)$, thus the radial variable appropriate for its description is

$$z = r - \rho(t) = O(1).$$

Define

$$R(t) = R(\epsilon^{-1/\alpha}\tilde{t}) = \tilde{R}(\tilde{t}),$$

$$u(r,t) = u\left(z + \epsilon^{-1}\tilde{R}(\tilde{t}), \epsilon^{-1/\alpha}\tilde{t}\right) = \tilde{u}(z,\tilde{t}).$$

Later on, we drop the tildes. One can find that

$$\frac{1}{r} = \frac{\epsilon}{R(t)} - \frac{z\epsilon^2}{R^2(t)} + \cdots, \tag{6}$$

hence we can rewrite Eq. (4) as

$$\partial_z^2 u + f(u) = \epsilon\partial_t^\alpha u - \frac{\epsilon}{R(t)}\partial_z u + O(\epsilon^2).$$

Because the motion of the front is caused solely by its curvature, the term

$$\partial_t^\alpha u(z,t) = \frac{-1}{\Gamma(-\alpha)}\int_0^\infty \left[u(z,t) - u\Big(z + \rho(t) - \rho(t-\tau), t\Big)\right]\frac{d\tau}{\tau^{\alpha+1}},$$

should balance the curvature term $\partial_z u/R(t)$; thus, both of them are of the same order. That justifies the choice of the time scaling (5).

3.1. Governing equation for a circular front

Using the expansion

$$u = u_0 + \epsilon u_1 + \cdots, \tag{7}$$

we obtain, at the leading order, the equation,

$$\partial_z^2 u_0 + f(u_0) = 0, \tag{8}$$

which describes the structure of the transient zone (domain wall, "kink"), $u_0(z) = u_f(z)$. In the case $f(u_0) = u_0 - u_0^3$, the kink solution of (8) is

$$u_f(z) = \tanh\left(\frac{z}{\sqrt{2}}\right); \tag{9}$$

in the case $f(u_0) = \text{sign}(u_0)(1 - |u_0|)$, we obtain

$$u_f(z) = \text{sign}(z)\left(1 - e^{-|z|}\right).$$

At the first order in ϵ, we search for a bounded solution of the equation,

$$\partial_z^2 u_1 + f'\left(u_f(z)\right)u_1 = \partial_t^\alpha u_f(z) - \frac{1}{R(t)}\partial_z u_f(z). \tag{10}$$

Because the operator in the left-hand side of (10) is self-adjoint and has a bounded homogeneous solution $\partial_z u_f(z)$, thus the solvability condition is the orthogonality of the right-hand side of (10) to $\partial_z u_f(z)$, i.e.,

$$\int_{-\infty}^\infty dz\,\partial_z u_f(z)\partial_t^\alpha u_f(z) = \frac{1}{R(t)}\int_{-\infty}^\infty dz[\partial_z u_f(z)]^2. \tag{11}$$

The left-hand side of (11),

$$\frac{-1}{\Gamma(-\alpha)}\int_{-\infty}^\infty dz\,\partial_z u_f(z) \times \int_0^\infty \frac{d\tau}{\tau^{\alpha+1}}\left[u_f(z) - u_f\Big(z + \rho(t) - \rho(t-\tau)\Big)\right] = \frac{-1}{\Gamma(-\alpha)}\int_0^\infty \frac{d\tau}{\tau^{\alpha+1}}G\Big(\rho(t) - \rho(t-\tau)\Big), \tag{12}$$

where

$$G(s) = \frac{1}{2}\left(u_f^2(\infty) - u_f^2(-\infty)\right) - \int_{-\infty}^\infty dy\,\partial_y u_f(y)u_f(y+s). \tag{13}$$

We discuss the convergence of the integral in (12) in Appendix A. Returning to the original temporal scale, we obtain the governing equation for the front shape $\rho(t)$,

$$\frac{-1}{\Gamma(-\alpha)}\int_0^\infty \frac{d\tau}{\tau^{\alpha+1}}G\Big(\rho(t) - \rho(t-\tau)\Big) = \frac{1}{\rho(t)}\int_{-\infty}^\infty dz\left[\partial_z u_f(z)\right]^2. \tag{14}$$