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Stabilisation of difference equations with noisy prediction-based control

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HIGHLIGHTS

- Noise is introduced to a class of difference equations under prediction-based control.
- The class of equations includes common models of population dynamics.
- Noise in the control parameter improves our ability to stabilise positive equilibria.
- Systemic noise has a "blurring" effect on positive equilibria.

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ABSTRACT

We consider the influence of stochastic perturbations on stability of a unique positive equilibrium of a difference equation subject to prediction-based control. These perturbations may be multiplicative

$$x_{n+1} = f(x_n) - (\alpha + l\xi_{n+1})(f(x_n) - x_n), \quad n = 0, 1, \dots,$$

if they arise from stochastic variation of the control parameter, or additive

$$x_{n+1} = f(x_n) - \alpha(f(x_n) - x_n) + l\xi_{n+1}, \quad n = 0, 1, \dots,$$

if they reflect the presence of systemic noise.

We begin by relaxing the control parameter in the deterministic equation, and deriving a range of values for the parameter over which all solutions eventually enter an invariant interval. Then, by allowing the variation to be stochastic, we derive sufficient conditions (less restrictive than known ones for the unperturbed equation) under which the positive equilibrium will be globally a.s. asymptotically stable: i.e. the presence of noise improves the known effectiveness of prediction-based control. Finally, we show that systemic noise has a "blurring" effect on the positive equilibrium, which can be made arbitrarily small by controlling the noise intensity. Numerical examples illustrate our results.

1. Introduction

The dynamics of discrete maps can be complicated, and various methods may be introduced to control their asymptotic behaviour. In addition, both the intrinsic dynamics and the control may involve stochasticity.

We may ask the following of stochastically perturbed difference equations:

(1) If the original (non-stochastic) map has chaotic or unknown dynamics, can we stabilise the equation by introducing a control with a stochastic component?

(2) If the non-stochastic equation is either stable or has known dynamics (for example, a stable two-cycle [1]), do those dynamics persist when a stochastic perturbation is introduced?

In this article, we consider both these questions in the context of prediction-based control (PBC, or predictive control). Ushio and Yamamoto [2] introduced PBC as a method of stabilising unstable

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periodic orbits of

$$x_{n+1} = f(x_n), \quad x_0 > 0,\ n \in \mathbb{N}_0, \tag{1}$$

where $\mathbb{N}_0 = \{0, 1, 2, \ldots, \}$. The method overcomes some of the limitations of delayed feedback control (introduced by Pyragas [3]), and does not require the a priori approximation of periodic orbits, as does the OGY method developed by Ott et al. [4].

The general form of PBC is

$$x_{n+1} = f(x_n) - \alpha(f^k(x_n) - x_n), \quad x_0 > 0,\ n \in \mathbb{N}_0,$$

where $\alpha \in (0, 1)$ and f^k is the kth iteration of f. If $k = 1$, PBC becomes

$$x_{n+1} = f(x_n) - \alpha(f(x_n) - x_n) = (1-\alpha)f(x_n) + \alpha x_n, \quad x_0 > 0,\ n \in \mathbb{N}_0. \tag{2}$$

Recently, it has been shown how PBC can be used to manage population size via population reduction by ensuring that the positive equilibrium of a class of one-dimensional maps commonly used to model population dynamics is globally asymptotically stable after the application of the control [5]. Similar effects are also possible if it is not feasible to apply the control at every timestep. This variation on the technique is referred to as PBC-based pulse stabilisation [6,7].

Here, we investigate the influence of stochastic perturbations on the ability of PBC to induce global asymptotic stability of a positive point equilibrium of a class of equations of the form (1). It is reasonable to introduce noise in one of two ways. First, the implementation of PBC relies upon a controlling agent to change the state of the system in a way characterised by the value of the control parameter α. In reality we expect that such precise control is impossible, and the actual change will be characterised by a control sequence $\{\alpha_n\}_{n\in\mathbb{N}_0}$ with terms that vary randomly around α with some distribution. This will lead to a state-dependent, or multiplicative, stochastic perturbation. Second, the system itself may be subject to extrinsic noise, which may be modelled by a state-independent, or additive, perturbation.

The fact that stochastic perturbation can stabilise an unstable equilibrium has been understood since the 1950s: consider the well-known example of the pendulum of Kapica [8]. More recently, a general theory of stochastic stabilisation and destabilisation of ordinary differential equations has developed from [9]: a comprehensive review of the literature is presented in [10]. This theory extends to functional differential equations: for example [11,12] and references therein.

Stochastic stabilisation and destabilisation is also possible for difference equations; see for example [13,14]. However, the qualitative behaviour of stochastic difference equations may be dramatically different from that seen in the continuous-time case, and must be investigated separately. For example, in [15], solutions of a nonlinear stochastic difference equation with multiplicative noise arising from an Euler discretisation of an Itô-type SDE are shown to demonstrate monotonic convergence to a point equilibrium with high probability. This behaviour is not possible in the continuous-time limit.

Now, consider the structure of the map f. We impose the Lipschitz-type assumption on the function f around the unique positive equilibrium K.

Assumption 1.1. $f : [0,\infty) \to [0,\infty)$ is a continuous function, $f(x) > 0$ for $x > 0$, $f(x) > x$ for $x \in (0, K)$, $f(x) < x$ for $x > K$, and there exists $M \geq 1$ such that

$$|f(x) - K| \leq M|x - K|. \tag{3}$$

Note that under Assumption 1.1 function f has only a single positive point equilibrium K. We will also suppose that f is decreasing on an interval that includes K:

Assumption 1.2. There is a point $c < K$ such that $f(x)$ is monotone decreasing on $[c, \infty)$.

It is quite common for Assumptions 1.1 and 1.2 to hold for models of population dynamics, and in particular for models characterised by a unimodal map: we illustrate this with Examples 1.3–1.5. It follows from Singer [16] that, when additionally f has a negative Schwarzian derivative $(Sf)(x) = f'''(x)/f'(x) - \frac{3}{2}(f''(x)/f'(x))^2 < 0$, the equilibrium K is globally asymptotically stable if and only if it is locally asymptotically stable. In each case, as the system parameter grows, a stable cycle replaces a stable equilibrium which loses its stability, there are period-doubling bifurcations and eventually chaotic behaviour.

Example 1.3. For the Ricker model

$$x_{n+1} = x_n e^{r(1-x_n)}, \quad x_0 > 0,\ n \in \mathbb{N}_0, \tag{4}$$

Assumptions 1.1 and 1.2 both hold with $K = 1$, and the global maximum is attained at $c = 1/r < K = 1$ for $r > 1$. Let us note that for $r \leq 1$ the positive equilibrium is globally asymptotically stable and the convergence of solutions to K is monotone. However, for $r > 2$ the equilibrium becomes unstable.

Example 1.4. The truncated logistic model

$$x_{n+1} = \max\{rx_n(1-x_n), 0\}, \quad x_0 > 0,\ n \in \mathbb{N}_0, \tag{5}$$

with $r > 1$ and $c = \frac{1}{2} < K = 1 - 1/r$, also satisfies Assumptions 1.1 and 1.2. Again, for $r \leq 2$, the equilibrium K is globally asymptotically stable, with monotone convergence to K, while for $r > 3$ the equilibrium K is unstable.

Example 1.5. For the modifications of the Beverton–Holt equation

$$x_{n+1} = \frac{Ax_n}{1 + Bx_n^{\gamma}}, \quad A > 1,\ B > 0,\ \gamma > 1,\ x_0 > 0,\ n \in \mathbb{N}_0, \tag{6}$$

and

$$x_{n+1} = \frac{Ax_n}{(1 + Bx_n)^{\gamma}}, \quad A > 1,\ B > 0,\ \gamma > 1,\ x_0 > 0,\ n \in \mathbb{N}_0, \tag{7}$$

Assumption 1.1 holds. Also, (6) and (7) satisfy Assumption 1.2 as long as the point at which the map on the right-hand side takes its maximum value is less than that of the point equilibrium. If Assumption 1.2 is not satisfied, the function is monotone increasing up to the unique positive point equilibrium, and thus all solutions converge to the positive equilibrium, and the convergence is monotone. If all $x_n > K$, we have a monotonically decreasing sequence. If we fix B in (6) and (7) and consider the growing A, the equation loses stability and experiences transition to chaos through a series of period-doubling bifurcations.

The article has the following structure. In Section 2 we relax the control parameter α, replacing it with the variable control sequence $\{\alpha_n\}_{n\in\mathbb{N}_0}$, and yielding the equation

$$x_{n+1} = f(x_n) - \alpha_n(f(x_n) - x_n) = (1-\alpha_n)f(x_n) + \alpha_n x_n, \quad x_0 > 0,\ n \in \mathbb{N}_0. \tag{8}$$

We identify a range over which $\{\alpha_n\}_{n\in\mathbb{N}_0}$ may vary deterministically while still ensuring the global asymptotic stability of the positive equilibrium K. We confirm that, without imposing any constraints on the range of values over which the control sequence $\{\alpha_n\}_{n\in\mathbb{N}_0}$ may vary, there exists an invariant interval, containing K, under the controlled map. We then introduce constraints on terms of the sequence $\{\alpha_n\}_{n\in\mathbb{N}_0}$ which ensure that all solutions will eventually enter this invariant interval.

In Section 3, we assume that the variation of α_n around α is bounded and stochastic, which results in a PBC equation with