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On degree–degree correlations in multilayer networks

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HIGHLIGHTS

- Generalizations of assortativity metrics for multilayer networks are provided.
- The metrics show that degree–degree correlations should be measured system-wide.
- The new tools are applied to study disease spreading on multilayer networks.

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ABSTRACT

We propose a generalization of the concept of assortativity based on the tensorial representation of multilayer networks, covering the definitions given in terms of Pearson and Spearman coefficients. Our approach can also be applied to weighted networks and provides information about correlations considering pairs of layers. By analyzing the multilayer representation of the airport transportation network, we show that contrasting results are obtained when the layers are analyzed independently or as an interconnected system. Finally, we study the impact of the level of assortativity and heterogeneity between layers on the spreading of diseases. Our results highlight the need of studying degree–degree correlations on multilayer systems, instead of on aggregated networks.

1. Introduction

The use of network science to study the structure and dynamics of complex systems has proved to be a successful approach to understand the organization and function of several natural and artificial systems [1–4]. The traditional framework used up to a few years ago represents the structure of complex systems as single-layer (also referred to as monoplex) networks, in which only one type of connection is accounted for. However, this approach is limited because most natural and artificial systems such as the brain, our society or modern transportation networks [5,6], are made up by different constituents and/or different types of interaction. Indeed, their structure is organized in layers. For instance, in social networks individuals can be connected according to different social ties, such as friendship or family relationship (e.g. [7]). In transportation networks, routes of a single airline can be represented as a network, whose vertices (destinations) can be mapped into networks of several companies [8]. Gene co-expression networks consist of layers, each one representing a different signaling pathway or expression channel [9]. Therefore, mapping out the structure of these and similar systems as a monoplex network could lead to miss relevant information that could not be captured if the single layers are analyzed separately nor if all layers are collapsed altogether in an aggregated graph. Additionally, note that in most of these interconnected systems, the information travels not only among vertices of the same layer, but also between pairs of layers.

Recent advances in modeling the aforementioned systems include new mathematical formulations [10], the generalization of different metrics [6,10–12] and the impact of the multilayer structure on several dynamical processes [13,11,14–16]. Although clustering [12], centrality [11,17] and spectral properties [13,11,18] of multilayer networks have been addressed, a measure to quantify degree–degree correlations in multilayers is still lacking.

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Degree–degree correlations is a fundamental property of single-layer networks, impacting the spreading of diseases, synchronization phenomena and systems' resilience [19,3]. Additionally, it has been reported that different correlations arise in different kinds of networks: social networks are in general assortative, meaning that highly connected nodes tend to link with each other, whereas technological and biological systems have disassortative structures, in which high degree nodes are likely attached to low degree nodes [20].

For networks made up of more than one layer, only recently, Nicosia and Latora [21] considered the correlation between the degrees in two different layers. However, their methodology is only for node-aligned multiplex networks, which are special cases of multilayer networks (see [5]). In fact, multiplex networks are made up of N nodes that can be in one or more interacting layers. The links in each layer represent a given mode of interaction between the set of nodes belonging to that layer, whereas links connecting different layers stand for the different modes of interaction between objects involved in [5].

In this paper we study degree–degree correlations in multilayer systems and propose a way to generalize previous assortativity metrics by considering the tensorial formulation introduced in [10]. Our approach also covers a weighted version of assortativity [22] and the case in which the assortativity is given by the Spearman correlation coefficient, generalizing the definition in [23]. Aside from those, it worth mention the generalization for weighted and directed networks [24]. The study of a real dataset corresponding to the airport transportation network shows a contrasting behavior between the analyses of each layers independently and altogether, which reinforces the need for such a generalization of the assortativity measure. Finally, we study the influence of degree–degree correlations on epidemic spreading in multilayer networks. We verify that the impact of the disease depends on degree–degree correlations and also on the level of heterogeneity between the layers.

2. Assortativity in multilayer networks

Tensors are suitable for representation of multilayer networks. As showed in [10], tensors allow us to consider a branch of new relationships between nodes and layers, by encoding a multilayer network as a fourth order mixed tensor, $M^{\alpha\tilde{\delta}}_{\beta\tilde{\gamma}}$, i.e. 2-covariant and 2-contravariant basis, in the Euclidean space. Such representation is convenient for many operations, as discussed in [10]. We use the definition of the interlayer adjacency tensor $C^{\alpha}_{\beta}(\tilde{h}\tilde{r})$ that is a second order tensor which has the information of the relationships between nodes in layers $\tilde{h}$ and $\tilde{r}$. Note that $C^{\alpha}_{\beta}(\tilde{r}\tilde{r})$ is the adjacency matrix for the layer $\tilde{r}$ and belongs to $\mathbb{R}^{N\times N}$ space. Then, the multilayer adjacency tensor is expressed as the summation over all layers L of the tensorial product of the adjacency tensors, $C^{\alpha}_{\beta}(\tilde{h}\tilde{r})$, and the canonical Euclidean basis. Mathematically,

$$M^{\alpha\tilde{\gamma}}_{\beta\tilde{\delta}} = \sum_{\tilde{h},\tilde{r}}^{L} C^{\alpha}_{\beta}(\tilde{h}\tilde{r}) E^{\tilde{\delta}}_{\tilde{\gamma}}(\tilde{h}\tilde{r}) \tag{1}$$

which belongs to $\mathbb{R}^{N\times N\times L\times L}$ space.

Following Einstein's summation convention, the assortativity coefficient can be written as

$$\rho(\mathcal{W}^{\alpha}_{\beta}) = \frac{\mathcal{M}^{-1}\mathcal{W}^{\alpha}_{\beta}Q^{\beta}Q_{\alpha} - \left[1/2\mathcal{M}^{-1}\left(\mathcal{W}^{\alpha}_{\beta}Q_{\alpha}u^{\beta} + \mathcal{W}^{\alpha}_{\beta}Q^{\beta}u_{\alpha}\right)\right]^2}{\mathcal{M}^{-1}\left(\mathcal{W}^{\alpha}_{\beta}(Q_{\alpha})^2u^{\beta} + \mathcal{W}^{\alpha}_{\beta}(Q^{\beta})^2u_{\alpha}\right) - \left[1/2\mathcal{M}^{-1}\left(\mathcal{W}^{\alpha}_{\beta}Q_{\alpha}u^{\beta} + \mathcal{W}^{\alpha}_{\beta}Q^{\beta}u_{\alpha}\right)\right]^2} \tag{2}$$

where u is the 1-tensor, which is a tensor of rank 1 and has all elements equal to 1, $\mathcal{W}^{\alpha}_{\beta}$ is a second order tensor that summarizes the information that is being extracted and $\mathcal{M} = \mathcal{W}^{\alpha}_{\beta}U^{\beta}_{\alpha}$ is a normalization constant.

Let us explain in more details all terms appearing in the expression of $\rho(\mathcal{W}^{\alpha}_{\beta})$. First, we define

$$Q^{\alpha} = \mathcal{W}^{\alpha}_{\beta}u^{\beta}, \tag{3}$$

which is a 1-contravariant tensor and

$$Q_{\beta} = \mathcal{W}^{\alpha}_{\beta}u_{\alpha} \tag{4}$$

which is a 1-covariant tensor. Moreover, the indices are related to the direction of the relationships between nodes. Such a choice ensures a more general expression, capturing degree correlations on non-symmetric tensors and, consequently, in directed and weighted networks.

Due to the multiplex nature of such systems we obtain different types of correlations, which can be uncovered by operating on the adjacency tensor. First of all, it is possible to extract a single layer by the operation called *single layer extraction* [10]. In this case, the adjacency tensor is defined as

$$\mathcal{W}^{\alpha}_{\beta} = C^{\alpha}_{\beta}(\tilde{r}\tilde{r}) = M^{\alpha\tilde{\gamma}}_{\beta\tilde{\delta}}E^{\tilde{\delta}}_{\tilde{\gamma}}(\tilde{r}\tilde{r}), \tag{5}$$

which is a simple projection on the canonical basis, $E^{\tilde{\delta}}_{\tilde{\gamma}}(\tilde{r}\tilde{r})$. It is noteworthy that the results obtained from this projection are the same as those obtained by considering the layer $\tilde{r}$ as a monoplex network and applying the traditional formulation of assortativity [20]. On the other hand, to consider all layers together, we can use the *projected network*, which is a weighted single-layer network. Formally it is given as

$$\mathcal{W}^{\alpha}_{\beta} = P^{\alpha}_{\beta} = M^{\alpha\tilde{\delta}}_{\beta\tilde{\gamma}}U^{\tilde{\gamma}}_{\tilde{\delta}}. \tag{6}$$

Note that the projection presents self-edges and, as argued in [10], it is different from a weighted monoplex network, since self-edges code for inter-layer couplings between different replica of the same object. Thus they have a different meaning with respect to other edges. A version of the projection without self-edges is called *overlay network* and is given as the contraction over the layers [10], i.e.,

$$\mathcal{W}^{\alpha}_{\beta} = O^{\alpha}_{\beta} = M^{\alpha\tilde{\gamma}}_{\beta\tilde{\gamma}}. \tag{7}$$

Observe that the overlay network does not consider the contribution of the interlayer connections, whereas the projection does. As we will see later, comparisons between the assortativity of those two different representations of the system reveal the key role of such inter-links.

In both cases, i.e., for the overlay and the projected networks, we extract degree–degree correlations. Nodes with similar degrees connected in the same or different layers contribute positively to the assortativity coefficient. On the other hand, the connections between hubs and low degree nodes in the same or different layers decrease the assortativity. Self-edges always increase the assortativity, which yields different values of assortativity for the overlay and the projected networks. This gives information on the nature of the coupling between different replicas of the same object among different layers.

In some applications, it is interesting to calculate a pair-wise correlation between a set of nodes, for instance, between couple of layers. Thus, we propose a new operation, that we call *selection*, which is a projection over a selected set of layers:

$$\mathcal{W}^{\alpha}_{\beta}(\mathcal{L}) = S^{\alpha}_{\beta}(\mathcal{L}) = M^{\alpha\tilde{\delta}}_{\beta\tilde{\gamma}}\Omega^{\tilde{\gamma}}_{\tilde{\delta}}(\mathcal{L}), \tag{8}$$