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Consensus dynamics on random rectangular graphs

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HIGHLIGHTS

- The consensus dynamics model is applied to a random rectangular graph (RRG).
- An RRG generalizes the random geometric graph by embedding the nodes in a unit rectangle.
- Bounds for the diameter and the algebraic connectivity of RRG are obtained.
- It is proved that when the rectangle is elongated the RRGs become 'large worlds' with poor connectivity.
- It is proved that when the rectangle is elongated the time for consensus in the RRG grows to infinity.

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ABSTRACT

A *random rectangular graph* (RRG) is a generalization of the random geometric graph (RGG) in which the nodes are embedded into a rectangle with side lengths a and $b = 1/a$, instead of on a unit square $[0, 1]^2$. Two nodes are then connected if and only if they are separated at a Euclidean distance smaller than or equal to a certain threshold radius r. When $a = 1$ the RRG is identical to the RGG. Here we apply the consensus dynamics model to the RRG. Our main result is a lower bound for the time of consensus, i.e., the time at which the network reaches a global consensus state. To prove this result we need first to find an upper bound for the algebraic connectivity of the RRG, i.e., the second smallest eigenvalue of the combinatorial Laplacian of the graph. This bound is based on a tight lower bound found for the graph diameter. Our results prove that as the rectangle in which the nodes are embedded becomes more elongated, the RRG becomes a 'large-world', i.e., the diameter grows to infinity, and a poorly-connected graph, i.e., the algebraic connectivity decays to zero. The main consequence of these findings is the proof that the time of consensus in RRGs grows to infinity as the rectangle becomes more elongated. In closing, consensus dynamics in RRGs strongly depend on the geometric characteristics of the embedding space, and reaching the consensus state becomes more difficult as the rectangle is more elongated.

1. Introduction

Many real-world networked systems are embedded into geometrical spaces. These spatial networks, as they are known, may represent many different kinds of scenarios [1]. For instance, in urban street networks the nodes describe the intersection of streets, which are represented by the edges of the graph. These streets and their intersections are embedded in the two-dimensional space representing the surface occupied by the corresponding city. Similar situations occur with infrastructural and transportation systems ranging from water supply networks and railroads to the internet and wireless sensor networks (WSNs). In WSNs [2], the nodes represent the sensors which are deployed on a given geographical region and their communication defines the connectivity of the nodes. This is analogous to many other communication systems ranging from mobile phones to radio signals. On a different scale we can mention the vascular and cellular networks of nodes embedded into cells and biological tissues [3]; protein residue networks [3]; the networks of channels in fractured rocks [4]; the networks representing the corridors and galleries in animal nests [5,6]; and landscape networks [7], among others. For modeling these spatial networks it is necessary to have a theoretical model that captures both the topological features typical of complex networks and the spatial embedding of these specific kinds of systems. The most commonly used model for spatial networks is the so-called *random geometric graph* (RGG) [8–11]. In RGGs each node

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is randomly assigned geometric coordinates and then two nodes are connected if the (Euclidean) distance between them is smaller than or equal to a certain threshold r.

The RGG model has been widely used in the study of wireless sensor networks (WSNs) and peer-to-peer networks [12–14], where the problem of consensus has received great attention due to the fact that it allows the achieving of tasks with a minimum overhead of communication [15–19]. In the consensus protocols, as they are known in technological applications, the problem consists of making the scalar states of a set of agents converge to the same value under local communication constraints [20,21]. Thus, since the communication requires only local information there is no congestion due to network traffic. RGGs are also used to model populations which are geographically constrained in a certain region, like a city. This scenario is important, for instance, for the analysis of epidemic spreading in such populations [22,17,23,24]. In this sense Riley et al. [25] have remarked that RGGs "*provide a nice way of escaping the lack of local correlation and clustering that are implicit properties of the configuration graphs often used to explore epidemic dynamics*". In a similar fashion, RGGs can be used to model structured populations in which opinions, instead of viruses, are propagated. In this case the RGGs also capture very well the geographic constraints of the population and, in comparison with other models [26], they "*are more realistic for a number of reasons: (i) RGG is isotropic (on average) while regular lattice is not; (ii) the average degree for an RGG can be set to an arbitrary positive number, instead of a small fixed number for the lattice; (iii) RGGs closely capture the topology of random networks of short-range-connected spatially-embedded artificial agents*".

In the formulation of the RGG model it is assumed that the nodes are uniformly and independently distributed on a unit square (or a higher dimensional hypercube in the general case) [8,9]. This unit square represents the area on which the agents are interacting to reach a consensus state, and it could be a workplace, a city, or a forest, just to mention some examples. Such a square-like area is typical of many real-world scenarios. For instance, the city of San Francisco (USA) is known as the "*seven-by-seven-mile square*", due to the fact that the mainland part of the city is a square of nearly 11 km by 11 km. However, if we consider other cities, like Manhattan, the picture looks very different. Manhattan is 13.4 miles (21.6 km) long and 2.3 miles (3.7 km) wide, which resembles a rectangular shape instead of a square one. Based on this necessity of considering the influence of the rectangular shape on the topological and dynamical properties of the random networks deployed on these areas we have recently introduced the random rectangular graph (RRG) model [27]. In this case, the nodes are uniformly and independently distributed on a unit rectangle of given side lengths. When both sides are of the same length we recover the RGG in such a way that the RRG model generalizes the RGG one.

Here, we are interested in investigating analytically and computationally how the elongation of the rectangle in the RRG affects the consensus dynamics taking place on the nodes and edges of the networks constructed on them. We start by introducing the concept of the random rectangular graph (RRG), and continue with the description of the consensus model to be considered. Then, we state the main result of this work which proves that for a RRG with a fixed number of nodes and a given connection radius, the time for reaching consensus grows to infinity when the rectangle is very elongated. We finally support our analytic results with computational simulations for RRGs.

2. Preliminaries

Here we present some definitions, notations, and properties which will be used in this work (see [3]). For the basic definitions about networks the reader is directed to the literature (see for instance [3]). The notation used here is standard. For instance, k_i designate the degree of the node i. The matrix $K = \operatorname{diag}(k_i)$ designates the degree matrix of the graph and the matrix $\mathcal{L} = K - A$ is the graph Laplacian, where A stands for the adjacency matrix of the graph. It has entries

$$\mathcal{L}_{uv} = \begin{cases} k_i & \text{if } u = v \\ -1 & \text{if } (u, v) \in E \\ 0 & \text{otherwise} \end{cases} \quad \forall u, v \in V.$$

The eigenvalues of the Laplacian matrix are denoted here by: $0 = \mu_1 \le \mu_2 \le \cdots \le \mu_n$. If the network is connected the multiplicity of the zero eigenvalue is equal to one, i.e., $0 = \mu_1 < \mu_2 \le \cdots \le \mu_n$ and the smallest nontrivial eigenvalue μ_2 is known as the *algebraic connectivity* of graph [28,29]. Let U be the matrix of orthonormalized eigenvectors $\vec{\psi}_j$ of $\mathcal{L}$, i.e., $U = [\vec{\psi}_1 \ \cdots \ \vec{\psi}_n]$. The eigenvector $\vec{\psi}_2$ associated with the algebraic connectivity is known as the Fiedler vector [28]. Let Λ be the diagonal matrix of eigenvalues of the Laplacian matrix. Then, $\mathcal{L} = U \Lambda U^T$.

2.1. Random rectangular graphs

The RGG is defined by distributing uniformly and independently n points in the unit d-dimensional cube $[0, 1]^d$ [8]. Then, two points are connected by an edge if their Euclidean distance is at most r, which is a given fixed number known as the *connection radius*. That is, we create a disk of radius r centered at each node, and every node inside that disk is connected to the central node. This disk plays the role of the area of influence of a given node, such as the area of coverage of a mobile or wireless sensor.

In [27] we have considered a unit hyperrectangle as the Cartesian product $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$ where $a_i, b_i \in \mathbb{R}$, $a_i \le b_i$, and $1 \le i \le d$ instead of the unit square of the RGG. Hereafter we will restrict ourselves to the 2-dimensional case, which corresponds to a rectangle of unit area. Now, the RRG has been defined by distributing uniformly and independently n points in the unit rectangle $[a, b]$ and then connecting two points by an edge if their Euclidean distance is at most r. The rest of the construction process remains the same as for the RGG. This implies that $RRG \to RGG$ as $(a/b) \to 1$ and consequently the RRG is a generalization of the RGG.

In Fig. 1 we illustrate two RRGs with different values of the rectangle side length a and the same number of nodes and edges. In the first case when $a = 1$ the graph corresponds to the classical random geometric graph in which the nodes are embedded into a unit square. The second case corresponds to $a = 2$ and it represents a slightly elongated rectangle.

A few important structural parameters of RRGs have been determined analytically in a previous work by the current authors (see [27]). They include the average degree, the probability that the graphs are connected, their degree distributions, average path length and clustering coefficient.

2.2. Consensus dynamics on graphs

Let us consider that the state of the nodes of the graph at time t is stored in the vector $\vec{u}(t)$. Then, the variation of the state of the node i with time is controlled by the equation [21,20]:

$$\dot{\vec{u}}_i(t) = \sum_{(i,j)\in E} \left(\vec{u}_j(t) - \vec{u}_i(t)\right), \quad i = 1, 2, \ldots, n, \qquad (1)$$

which, for the kind of graphs we analyze in this work can be written as

$$\dot{\vec{u}}_i(t) = -\sum_{j=1}^{n} a_{ij} \left(\vec{u}_i(t) - \vec{u}_j(t)\right), \quad i = 1, 2, \ldots, n. \qquad (2)$$