Physica D 323–324 (2016) 40–48

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Erosion of synchronization: Coupling heterogeneity and network structure

Per Sebastian Skardal [a,b,*], Dane Taylor [c], Jie Sun [d], Alex Arenas [b]

[a] Department of Mathematics, Trinity College, Hartford, CT 06106, USA
[b] Departament d'Enginyeria Informatica i Matemátiques, Universitat Rovira i Virgili, 43007 Tarragona, Spain
[c] Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599, USA
[d] Department of Mathematics, Clarkson University, Potsdam, NY 13699, USA

HIGHLIGHTS

- Synchronization of networks.
- Erosion of synchronization.
- Network heterogeneity.

ARTICLE INFO

Article history:
Received 2 June 2015
Received in revised form
22 October 2015
Accepted 23 October 2015
Available online 1 November 2015

Keywords:
Synchronization
Complex networks
Nonlinear dynamics

ABSTRACT

We study the dynamics of network-coupled phase oscillators in the presence of coupling frustration. It was recently demonstrated that in heterogeneous network topologies, the presence of coupling frustration causes perfect phase synchronization to become unattainable even in the limit of infinite coupling strength. Here, we consider the important case of heterogeneous coupling functions and extend previous results by deriving analytical predictions for the total erosion of synchronization. Our analytical results are given in terms of basic quantities related to the network structure and coupling frustration. In addition to fully heterogeneous coupling, where each individual interaction is allowed to be distinct, we also consider partially heterogeneous coupling and homogeneous coupling in which the coupling functions are either unique to each oscillator or identical for all network interactions, respectively. We demonstrate the validity of our theory with numerical simulations of multiple network models, and highlight the interesting effects that various coupling choices and network models have on the total erosion of synchronization. Finally, we consider some special network structures with well-known spectral properties, which allows us to derive further analytical results.

1. Introduction

Self-organization and emergent collective behavior represent universal concepts that are vital in many nonlinear processes [1,2]. Synchronization of large ensembles of coupled oscillators plays a particularly important role in our understanding of complex and network-coupled dynamical systems [3,4]. Examples of the importance of synchronization can be found in natural phenomena, for instance the functionality of cardiac pacemakers [5], mammalian circadian rhythms [6], and rhythmic flashing of fireflies [7], as well as engineered systems, for instance arrays of Josephson junctions [8], the power grid [9], and pedestrian bridges [10]. A particularly useful model for studying the synchronization of nonidentical oscillators was developed by Kuramoto [11], who showed that under suitable conditions the dynamics of N coupled oscillators can be reduced to the dynamics of N phase angles θ_i, for $i = 1, \dots, N$. When placed on a network whose structure dictates the oscillators' interaction patterns, the evolution of each phase is given by

$$\dot{\theta}_i = \omega_i + K \sum_{j=1}^{N} A_{ij} H_{ij}(\theta_j - \theta_i), \tag{1}$$

where the natural frequency ω_i describes the preferred frequency of oscillator i in the absence of coupling, $K \geq 0$ is the global

* Corresponding author at: Department of Mathematics, Trinity College, Hartford, CT 06106, USA.
E-mail address: skardals@gmail.com (P.S. Skardal).

http://dx.doi.org/10.1016/j.physd.2015.10.015
0167-2789/

coupling strength, the adjacency matrix A_{ij} encodes the network interactions, which is assumed to be undirected such that $A_{ij} = A_{ji}$, and $H_{ij}(\theta)$ is the coupling function that describes the functional effect of oscillator j on oscillator i, which is assumed to be 2π-periodic and continuously differentiable.

The dynamics exhibited by Eq. (1) have been studied in various contexts [12–30] and have advanced our understanding of collective behavior, particularly regarding the interplay between structure and dynamics and their effects on synchronization. Typically, the extent of phase synchronization of the oscillators is measured by the classical Kuramoto order parameter r that is defined by [11]

$$re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}, \tag{2}$$

where the complex number $re^{i\psi}$ represents the oscillators' centroid in the complex unit circle. In particular, the order parameter r ranges from 0 to 1, indicating complete incoherence and perfect synchronization, respectively, while intermediate values typically correspond to partial synchronization. Alternatively, several studies have defined the degree of phase synchronization using a combination of the collection of local order parameters, defined $r_i e^{i\psi_i} = \sum_{j=1}^{N} A_{ij}e^{i\theta_j}$ for $i = 1, \ldots, N$ [14,24,28].

A key element of the model in Eq. (1) is the choice of coupling functions $H_{ij}(\theta)$ that defines the interactions between oscillators. For instance, the choice $H_{ij}(\theta) = \sin(\theta)$ yields the classical Kuramoto model [11], while the presence of additional modes can give rise to multi-branch entrainment, a.k.a. cluster synchronization [31–38]. Here, we focus our attention on systems with coupling frustration, as indicated by one or more non-zero values of the quantity

$$h_{ij} = H_{ij}(0)/\sqrt{2}H'_{ij}(0). \tag{3}$$

The physical interpretation of coupling frustration corresponds to the case where the networks' interaction terms do not all vanish when all phases are equal. The presence of coupling frustration is vital in the modeling of excitable and reaction–diffusion dynamics for the reason that neighboring elements typically do not react simultaneously, but rather one after another [39]. Many such examples exist in biological and chemical systems, including neuron excitation [40], cardiac dynamics [41], and the Belousov–Zhabotinsky reaction [42]. Additionally, coupling frustration has been linked to the emergence of chimera states [43–51], non-universal synchronization transitions [52], and other effects [53].

In a recent publication [54] we reported a novel phenomenon for networks of coupled oscillators that we called *erosion of synchronization*. In particular, we found that in the presence of both coupling frustration and structural heterogeneity the perfectly synchronized state (i.e., $r = 1$, or equivalently, $\theta_1 = \theta_2 = \cdots = \theta_N$) becomes unattainable in steady-state even in the limit of infinite coupling strength. To quantify the total erosion of synchronization in a network, we consider the quantity $1 - r$ in the limit $K \to \infty$, denoted $1-r^{\infty}$. We demonstrated this by considering the case of *homogeneous coupling*, i.e., $H_{ij}(\theta) = H(\theta)$, and subsequently showed that the total erosion of synchronization could be separated into the product of two terms describing the contributions of coupling frustration and structural heterogeneity, respectively, and that both of these terms amplify the total erosion of synchronization.

In this paper, we provide a more complete description of this phenomenon. In particular, we extend our previous results to account for the important case of heterogeneous coupling, i.e., when the coupling function governing the interaction between each pair of network neighbors may be distinct. We refer to this most general case, where each $H_{ij}(\theta)$ is potentially different, as *full coupling heterogeneity*. In this case we assume that each undirected link has an associated coupling function, so that $H_{ij}(\theta) = H_{ji}(\theta)$. We also treat the case where each oscillator has its own coupling function, i.e., $H_{ij}(\theta) = H_i(\theta)$, which we refer to as *partial coupling heterogeneity*. Unlike the homogeneous coupling case, in both the fully and partially heterogeneous coupling cases we find that the total erosion of synchronization cannot be separated into a product of contributions from the coupling frustration and structural heterogeneity.

The remainder of this paper is organized as follows. In Section 2 we present our theoretical results, which extend previous results for homogeneous coupling to the cases of both full and partial coupling heterogeneity. In Section 3 we present results from numerical simulations that support our theory and explore the interplay between coupling frustration and structural heterogeneity. In Section 4 we study the stability of the synchronized state. In Section 5 we investigate erosion of synchronization in several network models with well-known spectral properties, allowing us to develop further analytical results. In particular, we consider the star and chain networks, as well as Watts–Strogatz networks [55]. Finally, in Section 6 we conclude with a discussion of our results.

2. Theory

In this section we present a theoretical framework for quantifying the erosion of synchronization for the dynamics defined in Eq. (1). We begin by considering the case of fully heterogeneous coupling, i.e., where each undirected link connecting oscillators i and j have a potentially distinct coupling function $H_{ij}(\theta)$. We note that in this case we assume that the coupling is structurally symmetric, i.e., $H_{ij}(\theta) = H_{ji}(\theta)$, however no symmetry conditions are put on the functions $H_{ij}(\theta)$. In particular, $H_{ij}(\theta)$ need not be antisymmetric in θ, so that in general $H_{ij}(\theta)$ is not necessarily equal to $-H_{ij}(-\theta)$. We also consider the case of partially heterogeneous coupling, i.e., when each oscillator has its own coupling functions, $H_{ij}(\theta) = H_i(\theta)$. Finally, we compare these results to the originally derived results for homogeneous coupling, i.e., $H_{ij}(\theta) = H(\theta)$, presented in Ref. [54].

2.1. Fully heterogeneous coupling

We begin by following Ref. [27] and consider the dynamics of Eq. (1) in the strong coupling regime, i.e., $r \approx 1$. In typical networks, such a state can be attained in a variety of ways, most readily by considering either a sufficiently large coupling strength, or a set of natural frequencies with a sufficiently small spread. It is worth pointing out that these two situations are equivalent up to a rescaling of time, and thus the results presented here are valid in both cases. In the strong coupling regime the oscillators become tightly packed around the mean phase ψ, implying that $|\theta_i - \theta_j| \ll 1$ for all (i, j) pairs. Thus, the contribution of each pair-wise interaction can be linearized to $H_{ij}(\theta_j - \theta_i) \approx H_{ij}(0) + H'_{ij}(0)(\theta_j - \theta_i)$, and Eq. (1) can be approximated by

$$\dot{\theta}_i \approx \omega_i + K\tilde{d}_i - K\sum_{j=1}^{N}\tilde{L}_{ij}\theta_j, \tag{4}$$

or rather in vector form,

$$\dot{\boldsymbol{\theta}} \approx \boldsymbol{\omega} + K\tilde{\boldsymbol{d}} - K\tilde{L}\boldsymbol{\theta}. \tag{5}$$

Here, $\tilde{\boldsymbol{d}}$ and $\tilde{L}$ represent the *weighted degree vector* and *weighted Laplacian matrix*. In contrast to the unweighted degree vector $\boldsymbol{d}$ and